Let $\mathbb{N}_0 = \{0, 1, \ldots \}$ be the set of nonnegative integers and let $\mathcal{N} = \{i_1, \ldots, i_N\}$ be a fixed subset of $\mathbb{N}_0$. Then we denote by $\mathcal{B} = \mathcal{B}_\mathcal{N} = \mathbb{C}\langle e_{i_1}, \ldots, e_{i_N} \rangle$ the free unital associative $\mathbb{C}$-algebra with $N$ generators $\{e_i\}_{i \in \mathcal{N}}$, each of degree one. We can think of $\mathcal{B}$ as an algebra of noncommutative polynomials in $N$ noncommuting variables $e_{i_1}, \ldots, e_{i_N}$.

We equip $\mathcal{B}$ with a multiparametric $q_{ij}$-differential structure given by $N$ linear operators $\partial_i : \mathcal{B} \to \mathcal{B}$, $i \in \mathcal{N}$ that act as twisted derivations on $\mathcal{B}$:

\[
\partial_i(1) = 0, \quad \partial_i(e_j) = \delta_{ij}, \quad \partial_i(e_j x) = \delta_{ij} x + q_{ij} e_j \partial_i(x) \quad \text{for all } x \in \mathcal{B}, \ i, j \in \mathcal{N} (q_{ij} \text{ are complex numbers}).
\]

The algebra $\mathcal{B}$ is naturally graded by total degree $\mathcal{B} = \bigoplus_{n \geq 0} \mathcal{B}_{\leq n}$, where $\mathcal{B}_0 = \mathbb{C}$ and $\mathcal{B}_n$ consists of all homogeneous noncommuting polynomials of total degree $n$ in variables $e_{i_1}, \ldots, e_{i_N}$. More generally we also have a finer decomposition of $\mathcal{B}$ into multigraded components (= weight subspaces)

\[
\mathcal{B} = \bigoplus_{n \geq 0, t_1 \leq \cdots \leq t_n, t_j \in \mathcal{N}} \mathcal{B}_{t_1 \cdots t_n},
\]

where each weight subspace $\mathcal{B}_Q = \mathcal{B}_{t_1 \cdots t_n}$, corresponds to a multiset $Q = (l_1 \ldots l_n)$, is given by

\[
\mathcal{B}_Q = \text{span}_\mathbb{C} \left\{ e_{j_1 \cdots j_n} := e_{j_1} \cdots e_{j_n} \mid j_1 \cdots j_n \in \hat{Q} \right\}.
\]

Here $\hat{Q} = S_n Q = \{ \sigma(l_1 \ldots l_n) \mid \sigma \in S_n \}$ denotes the set of all rearrangements of the sequence $l_1, \ldots, l_n$ (i.e $\hat{Q}$ is the set of all distinct permutations of the multiset $Q$). Thus $\dim \mathcal{B}_Q = |\hat{Q}|$.

Of particular interest in algebra $\mathcal{B}$ are elements called constants which satisfy $\partial_i C = 0$ for every $i \in \mathcal{N}$. Let $\mathcal{C}$ denotes the space of all constants in algebra $\mathcal{B}$ and similarly let $\mathcal{C}_Q$ denotes the space of all constants in $\mathcal{B}_Q$. Then the main problem of describing the space $\mathcal{C}$ can be reduced to describing the space $\mathcal{C}_Q$. Here we shall give the explicit formulas for nontrivial (basic) constants in $\mathcal{B}_Q$ up to total degree equal to four.

MSC2010: 05Exx.

Keywords: $q$-algebras, noncommutative polynomial algebras, twisted derivations.

Section: 14.
References


