THE KERNEL OF MULTIDEGREE OPERATOR ON GENERIC SUBSPACES OF ALGEBRA $B$

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Let \( \mathcal{N} = \{i_1, \ldots, i_N\} \subseteq \{0, 1, \ldots\}, \quad \mathcal{B} = \mathbb{C} \langle e_{i_1}, \ldots, e_{i_N} \rangle \)

(deg \( e_i = 1, \forall i \in \mathcal{N} \)). The \( \mathbf{q} \)-differential structure on \( \mathcal{B} \) is given by \( \mathbf{q} \)-differential operators \( \partial_i = \partial_i : \mathcal{B} \to \mathcal{B}, i \in \mathcal{N} \) (of degree \(-1\))

twisted Leibnitz rule

\[
\partial_i(e_j x) = \delta_{ij} x + q_{ij} e_j \partial_i(x) \quad \forall x \in \mathcal{B}, i, j \in \mathcal{N}
\]

with \( \partial_i(1) = 0, \quad \partial_i(e_j) = \delta_{ij}, \) \( (\delta_{ij} = \text{standard Kronecker delta}) \).

The algebra \( \mathcal{B} \) is graded by total degree \( \mathcal{B} = \bigoplus \mathcal{B}^n \); more generally it is multigraded:

\[
\mathcal{B} = \bigoplus_{n \geq 0, l_1 \leq \cdots \leq l_n, l_j \in \mathcal{N}} \mathcal{B}_{l_1 \cdots l_n}
\]

where every weight subspace (multigraded component)

\[
\mathcal{B}_Q = \text{span}_{\mathbb{C}} \left\{ e_{j_1} \cdots j_n \equiv e_{j_1} \cdots e_{j_n} \mid j_1 \cdots j_n \in \hat{Q} \right\}
\]

corresponds to a multiset \( Q = \{l_1 \leq \cdots \leq l_n\} \) over \( \mathcal{N} \) of size \( n = \text{Card} \ Q \).

\( \hat{Q} \) = the set of all distinct permutations of \( Q \). Thus \( \text{dim} \mathcal{B}_Q = \text{Card} \ \hat{Q} \).
Let \( \underline{j} := j_1 \ldots j_n \) and let \( \mathfrak{B}_Q = \{ e_{\underline{j}} | \underline{j} \in \hat{Q} \} \) denote the monomial basis of \( \mathcal{B}_Q \).

Then the action of \( \partial_i \) on \( e_{\underline{j}} \in \mathfrak{B}_Q \) is given explicitly by:

\[
\partial_i(e_{\underline{j}}) = \sum_{1 \leq k \leq n, j_k = i} q_{ij_1} \cdots q_{ij_{k-1}} e_{j_1\ldots \hat{j}_k\ldots j_n},
\]

(1)

(\( \hat{j}_k \) denotes the omission of \( j_k \)).

**Example**

\[
\partial_2(e_{231242}) = e_{31242} + q_{22}q_{23}q_{21} e_{23142} + q_{22}^2q_{23}q_{21}q_{24} e_{23124}.
\]

Note: the number of terms in this sum is equal to the number of appearances of the generator \( e_2 \) in monomial \( e_{231242} = e_2e_3e_1e_2e_4e_2 \).

Here we have marked each passing of \( \partial_2 \) through \( e_{231242} \) from the left by corresponding additional factors \( q_{2j}'s \).

- If \( Q \) is a set (sometimes called the generic case), then the formula (1) is reduced to:

\[
\partial_{j_k}(e_{\underline{j}}) = q_{j_kj_1} \cdots q_{j_kj_{k-1}} e_{j_1\ldots \hat{j}_k\ldots j_n}.
\]

(2)
With the motivation of treating better the matrices of $\partial_i |_{B_Q}$, we introduce a multidegree operator $\partial : B \to B$, $\partial = \sum_{i \in \mathbb{N}} e_i \partial_i$ and denote by $\partial^Q = \partial^Q |_{B_Q} : B_Q \to B_Q$ the restriction of $\partial$ to the subspace $B_Q$. Then for each $e_j \in B_Q$ we get:

$$\partial^Q(e_j) = \sum_{1 \leq k \leq n} q_{j_k j_1} \cdots q_{j_k j_{k-1}} e_{j_k j_1 \cdots \hat{j}_k \cdots j_n}$$  \hspace{1cm}(3)

i.e

$$\partial^Q(e_j) = e_{j_1 \cdots j_n} + q_{j_2 j_1} e_{j_2 j_1 \cdots j_n} + \cdots + q_{j_n j_1} \cdots q_{j_n j_{n-1}} e_{j_n j_1 \cdots j_{n-1}}$$

Let $B_Q :=$ the matrix of $\partial^Q$ w.r.t basis $B_Q$ of $B_Q$. Then $B_Q$ is the square matrix and its size is equal to $\dim B_Q (= \text{Card } \widehat{Q})$. The entries of $B_Q$ are polynomials in $q_{ij}$'s, that are reduced to monomials in the generic case (i.e $Q$ is a set) where we have got:

$$\det B_Q = \prod_{\substack{T \subseteq Q \\ 2 \leq |T| \leq n}} (1 - \sigma_T)^{|T| - 2!(n-|T|)!} \quad \text{with} \quad \sigma_T = \prod_{a \neq b \in T} q_{ab}, \ (|T| = \text{Card } T)$$

- We say that $q_{ij}$'s are singular parameters if they annihilate $\det B_Q$; otherwise they are in general position.
- Of particular interest $Q$-cocycle condition: $\sigma_Q = 1, \ \sigma_T \neq 1$ for all $T \subsetneq Q$ which we will write shorter: $\sigma_Q = 1$. 

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Recall, the formula (3): $\partial^Q(e_j) = \sum_{1 \leq k \leq n} q_{j_k j_1 \cdots j_k j_{k-1}} e_{j_k j_1 \cdots j_k \cdots j_n}$

Example

Let $Q = \{1, 2, 3\}$. Then $\partial^Q(e_{j_1 j_2 j_3}) = e_{j_1 j_2 j_3} + q_{j_2 j_1} e_{j_2 j_1 j_3} + q_{j_3 j_1} q_{j_3 j_2} e_{j_3 j_1 j_2}$ and the matrix $B_{123}$ of $\partial^Q$ w.r.t basis of $B_{123}$ is given by:

$$B_{123} = \begin{bmatrix}
1 & 0 & 0 & 0 & q_{12}q_{13} & q_{12} \\
0 & 1 & q_{13} & q_{13}q_{12} & 0 & 0 \\
q_{31}q_{32} & q_{31} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & q_{32} & q_{32}q_{31} \\
0 & 0 & q_{23}q_{21} & q_{23} & 1 & 0 \\
q_{21} & q_{21}q_{23} & 0 & 0 & 0 & 1 \\
\end{bmatrix}$$

where $\det B_{123} = (1 - \sigma_{12})(1 - \sigma_{13})(1 - \sigma_{23})(1 - \sigma_{123})$.

Then, the $Q$-cocycle condition is given by: $\sigma_{123} = 1$, $\sigma_{12} \neq 1$, $\sigma_{13} \neq 1$, $\sigma_{23} \neq 1$ or shorter: $\sigma_{123} = 1$

with $\sigma_{123} = \sigma_{12}\sigma_{13}\sigma_{23} = q_{12}q_{21}q_{13}q_{31}q_{23}q_{32}$. 
Definition

A constant \( C \in B \) is an element in \( B \) that satisfies \( \partial_i C = 0 \) (\( \forall i \in \mathbb{N} \)).

- Note:
  \[ \partial_i C = 0 \iff \partial C = 0 \]

\( \mathcal{C} \) = the space of all constants in the algebra \( B \); \( \mathcal{C}_Q \) = the space of all constants in \( B_Q \).

Thus \( \mathcal{C} = \ker \partial, \mathcal{C}_Q = \ker \partial^Q \).

- A fundamental problem: determine \( \ker \partial \) (i.e. describe the space \( \mathcal{C} \)).

By considering that \( \partial \) preserves the direct sum decomposition of \( B \), the fundamental problem can be reduced to determine finite dimensional \( \ker \partial^Q \) for every multiset \( Q \) over \( \mathbb{N} \).

- Our motivation: determine finite dimensional \( \ker \partial^Q \), when \( Q \) is a set

(a detailed examination of the constants in \( B_Q \) its leads us to the conclusion that the constants in degenerated \( B_Q \) can be constructed from those in the generic ones by a certain specialization procedure).

Recall, \( B_Q \) = generic subspace of \( B \) if \( Q \) is a set, otherwise \( B_Q \) we call degenerate.

- To solve this problem:
  show that \( \partial^Q \) can be factorized in terms of simpler operators,

  (one needs some special operators and their factorizations in terms of yet simpler operators). Recall, \( \partial^Q(e_j) = B(e_j) \).
A simpler approach:

1. study certain canonical elements in the twisted group algebra $\mathcal{A}(S_n)$;
2. use a natural representation of $\mathcal{A}(S_n)$ on the generic $\mathcal{B}_Q \subset \mathcal{B}$,

where some factorizations of certain canonical elements from $\mathcal{A}(S_n)$ will immediately give the corresponding matrix factorizations and also determinant factorizations.

Let $\mathcal{A}(S_n) = R_n \rtimes \mathbb{C}[S_n]$ denote a twisted group algebra of the symmetric group $S_n$ with coefficients in the polynomial ring $R_n = \mathbb{C}[X_{ab}, 1 \leq a, b \leq n]$ in $n^2$ commuting variables $X_{ab}$. The multiplication in $\mathcal{A}(S_n)$:

$$(p_1(\ldots, X_{ab}, \ldots) g_1) \cdot (p_2(\ldots, X_{cd}, \ldots) g_2) = p_1(\ldots, X_{ab}, \ldots) \cdot p_2(\ldots, X_{g_1(c) g_1(d)}, \ldots) g_1 g_2$$

- We will use that canonically defined elements in $\mathcal{A}(S_n)$:

$$\alpha_n^* = \sum_{g \in S_n} \left( \prod_{(a,b) \in I(g^{-1})} X_{ab} \right) g \left( \sum_{g \in S_n} g^* \right)$$

$I(g) = \{(a, b) \mid 1 \leq a < b \leq n, g(a) > g(b)\}$ (the set of all inversions $(a, b)$ of the permutation $g$)

- can be factorized by:

$$\alpha_n^* = \beta_1^* \cdot \beta_2^* \cdots \beta_n^*$$
\[ \alpha^*_n = \sum_{s \in S_n} \left( \prod_{(a,b) \in I(g^{-1})} X_{a,b} \right) g \quad \rightarrow \quad \alpha^*_n = \beta^*_1 \cdot \beta^*_2 \cdots \beta^*_n \]

where we define:

\[ \beta^*_{n-k+1} = t^*_{n,k} + t^*_{n-1,k} + \cdots + t^*_{k+1,k} + t^*_{k,k} \quad (1 \leq k \leq n) \]

with

\[ t^*_{b,a} = \left( \prod_{a+1 \leq j \leq b} X_{a,j} \right) t_{b,a} \quad \text{for each} \quad 1 \leq a \leq b \leq n. \]

Note \( t_{b,a} \) denotes the cycle \((a, a+1 \ldots, b)\). In particular, for \( k = n \) it follows: \( \beta^*_1 = \text{id} \).

\( \bullet \) If we define yet simpler elements in \( \mathcal{A}(S_n) \) for all \( 1 \leq k \leq n - 1 \)

\[ \gamma^*_{n-k+1} = (\text{id} - t^*_{n,k}) \cdot (\text{id} - t^*_{n-1,k}) \cdots (\text{id} - t^*_{k+1,k}), \]

\[ \delta^*_{n-k+1} = (\text{id} - (t^*_k)^2 t^*_{n,k+1}) \cdot (\text{id} - (t^*_k)^2 t^*_{n-1,k+1}) \cdots (\text{id} - (t^*_k)^2 t^*_{k+1,k+1}), \]

then we get further factorization:

\[ \beta^*_{n-k+1} = \delta^*_{n-k+1} \cdot (\gamma^*_{n-k+1})^{-1} \]

where we used:

\[ t^*_{k+1,k+1} = \text{id}, \quad (t^*_k)^2 = (X_{k,k+1} \cdot X_{k+1,k}) \text{id}. \]

Thus the canonically defined elements \( \alpha^*_n \in \mathcal{A}(S_n) \) can be factorized:

\[ \alpha^*_n = \beta^*_1 \cdot \beta^*_2 \cdots \beta^*_n \quad \text{with} \quad \beta^*_k = \delta^*_k \cdot (\gamma^*_k)^{-1} \quad (2 \leq k \leq n, \ \beta^*_1 = \text{id}). \]
Our next task is to define a representation \( \varphi: \mathcal{A}(S_n) \to \text{End}(\mathcal{B}_Q) \), where \( \mathcal{B}_Q = \text{span}_\mathbb{C} \left\{ e_j \mid j \in \hat{Q} \right\} \) is generic subspace of the algebra \( \mathcal{B} \).

Due \( \mathcal{A}(S_n) = R_n \rtimes \mathbb{C}[S_n] \) we consider a representation \( \varphi_1 \) of \( R_n \) and then a representation \( \varphi_2 \) of \( \mathbb{C}[S_n] \) as follows:

\[
\varphi_1: R_n \to \text{End}(\mathcal{B}_Q), \quad \varphi_1(X_{ab}) := Q_{ab} \quad (X_{ab} \in R_n, 1 \leq a, b \leq n),
\]

\[
\varphi_2: \mathbb{C}[S_n] \to \text{End}(\mathcal{B}_Q), \quad \varphi_2(g) e_{j_1 \ldots j_n} := e_{j_{g^{-1}(1)} \ldots j_{g^{-1}(n)}} \quad (\forall g \in S_n),
\]

\( Q_{ab} \) \((1 \leq a, b \leq n)\) denotes a diagonal operator on \( \mathcal{B}_Q \) defined: \( Q_{ab} e_j := q_{ja} q_{jb} e_j \).

**Theorem**

A map \( \varphi: \mathcal{A}(S_n) \to \text{End}(\mathcal{B}_Q) \) defined by \( \varphi(pg) := \varphi_1(p) \cdot \varphi_2(g), \) \((\forall p \in R_n, g \in S_n)\) is a representation.

To prove this Theorem it is enough to check that:

1. \( \varphi(X_{ab} \cdot X_{cd}) = \varphi(X_{cd} \cdot X_{ab}) \) \((\text{note that: diagonal operators commute})\)
2. \( \varphi(g \cdot X_{ab}) e_{j_1 \ldots j_n} = \varphi(X_{g(a) g(b)} g) e_{j_1 \ldots j_n} = q_{ja} q_{jb} e_{j_{g^{-1}(1)} \ldots j_{g^{-1}(n)}} \).
By considering above defined representations $\varrho_1, \varrho_2$ and a typical element $g^* \in A(S_n)$, 

$g^* = \left( \prod_{(a,b) \in I(g^{-1})} X_{a,b} \right) g$ \quad (recall: $\alpha_n^* = \sum_{g \in S_n} g^*$) it follows:

$\varrho(g^*) = \prod_{(a,b) \in I(g)} q_{jb} q_{ja} e_j g_{-1}(1) \cdots g_{-1}(n)$

and:

$\varrho(t_{b,a}^*) e_{j_1} \cdots j_{a} j_{a+1} \cdots j_{b} \cdots j_{n} = q_{jb} q_{ja} q_{jb} q_{ja+1} \cdots q_{jb} q_{ja-1} e_{j_1} \cdots j_{a} j_{a+1} \cdots j_{b} \cdots j_{n},$

$\varrho((t_a^*)^2) e_{j_1} \cdots j_{n} = \sigma_{j_a j_{a+1}} e_{j_1} \cdots j_{n}$ \quad (where $t_a^* = t_{a+1,a}^*$, $\sigma_{j_a j_{a+1}} = q_{ja} q_{ja+1} q_{ja+1} j_{a}$).

Then we get:

$\varrho(\beta_{n-k+1}^*) e_{j} = \sum_{1 \leq m \leq n} \varrho(t_{m,k}^*) e_{j}, \quad 1 \leq k \leq n$ \quad (with $\varrho(\beta_1^*) e_{j} = e_{j}$ if $k = 1$)

and in particular, for $k = 1$

$\varrho(\beta_{n}^*) e_{j} = \sum_{1 \leq m \leq n} \varrho(t_{m,1}^*) e_{j}$

with:

$\varrho(t_{m,1}^*) e_{j} = q_{jm} q_{jm} q_{jm} \cdots q_{jm} q_{jm} q_{jm-1} e_{jm} \cdots j_{m-1} j_{m+1} \cdots j_{n}$.

Then the last red formula can be rewritten, in the matrix notation, as follows:

$B_{Q,n} e_{j} = \sum_{1 \leq m \leq n} q_{jm} q_{jm} \cdots q_{jm} q_{jm} q_{jm-1} e_{jm} \cdots j_{m-1} j_{m+1} \cdots j_{n},$

where we have got $B_{Q,n} e_{j} = B_Q(e_{j})$ with: $B_Q(e_{j}) = \partial_Q(e_{j})$. Then we obtain:

$B_{Q,n} = D_{Q,n} \cdot (C_{Q,n})^{-1}$ \quad (where $C_{Q,n} := \varrho(\gamma^*_n)$, $D_{Q,n} := \varrho(\delta^*_n)$).

Now, in the operator notation, we can rewrite: $\partial_Q \cdot C_{Q,n} = D_{Q,n}$
\[ \partial^Q \cdot C_{Q,n} = D_{Q,n} \]

\[ C_{Q,n} = (id - T_{n,1}) \cdot (id - T_{n-1,1}) \cdots (id - T_{2,1}) \]

\[ D_{Q,n} = (id - (T_1)^2 T_{n,2}) \cdot (id - (T_1)^2 T_{n-1,2}) \cdots (id - (T_1)^2 T_{2,2}) \]

where

\[ T_{m,1} e_{\underline{j}} = q_{jm_1} \cdots q_{jm_{m-1}} e_{jm_1 \cdots j_{m-1}j_{m+1} \cdots j_n} \quad (T_{1,1} = id) \]

\[ (T_1)^2 T_{m,2} e_{\underline{j}} = \sigma_{j_1 j_m} q_{j_1 j_2} \cdots q_{j_{m-1} j_m} e_{j_1 j_2 \cdots j_{m-1}j_{m+1} \cdots j_n} \]

**Theorem**

Suppose that \( U_{\underline{j}} \in \ker (id - (T_1)^2 T_{n,2}) \) then the corresponding vector \( X_{\underline{j}} \in \ker \partial^Q \) is given by

\[ X_{\underline{j}} = C_{Q,n} \cdot \prod_{2 \leq m \leq n-1} (id - (T_1)^2 T_{m,2})^{-1} \cdot U_{\underline{j}}. \]

(4)

**Proof.**

Observe:

\[ \partial^Q \cdot C_{Q,n} = (id - (T_1)^2 T_{n,2}) \cdot \prod_{2 \leq m \leq n-1} (id - (T_1)^2 T_{m,2}) \quad \text{i.e.:} \]

\[ \partial^Q \cdot C_{Q,n} \cdot \prod_{2 \leq m \leq n-1} (id - (T_1)^2 T_{m,2})^{-1} \cdot U_{\underline{j}} = (id - (T_1)^2 T_{n,2}) \cdot U_{\underline{j}} \quad (\forall U_{\underline{j}} \in \mathcal{B}). \]

Then for each \( U_{\underline{j}} \in \ker (id - (T_1)^2 T_{n,2}) \) the right hand side of the last formula is equal to zero. Hence every \( X_{\underline{j}} \in \ker \partial^Q \) is given by (4).
Of particular interest are the vectors \( U_j \in \ker \left( id - (T_1)^2 T_{n,2} \right) \). By using:

\[
(id - (T_1)^2 T_{n,2})^{n-1} e_j = (1 - \sigma_Q) e_j,
\]

\[
(id - (T_1)^2 T_{n,2})^{n-1} e_j = (id - (T_1)^2 T_{n,2}) \cdot \left( id + ((T_1)^2 T_{n,2}) + \cdots + ((T_1)^2 T_{n,2})^{n-2} \right) e_j
\]

where \( \sigma_Q = \prod_{\{i,j\} \subset Q} \sigma_{ij} = \prod_{i \neq j \in Q} q_{ij} \), \((Q = \text{set})\), we get:

\[
(id - (T_1)^2 T_{n,2}) \cdot \left( id + ((T_1)^2 T_{n,2}) + \cdots + ((T_1)^2 T_{n,2})^{n-2} \right) e_j = (1 - \sigma_Q) e_j. \quad (5)
\]

- \( U_j \) belongs to \( \ker \left( id - (T_1)^2 T_{n,2} \right) \) if \( \sigma_Q = 1 \) (i.e \( Q \)-cocycle condition is satisfied).

\[
U_j := \left( id + ((T_1)^2 T_{n,2}) + \cdots + ((T_1)^2 T_{n,2})^{n-2} \right) e_j. \quad (6)
\]

By using: \( Q\{1,\ldots,n\} \) = a diagonal operator on \( B_Q \), from (6) and (5) it follows

\[
U_j = (id - (T_1)^2 T_{n,2})^{-1} \cdot (id - Q\{1,\ldots,n\}) e_j
\]

- Then \( X_j \in \ker \partial^Q \), \( X_j = C_{Q,n} \cdot \prod_{2 \leq m \leq n-1} (id - (T_1)^2 T_{m,2})^{-1} \cdot U_j \) is given by

\[
X_j = C_{Q,n} \cdot \prod_{2 \leq m \leq n} (id - (T_1)^2 T_{m,2})^{-1} \cdot (id - Q\{1,\ldots,n\}) e_j
\]

\[= (D_{Q,n})^{-1}\]
\[ X_j = C_{Q,n} \cdot (D_{Q,n})^{-1} \cdot (id - Q_{\{1,...,n\}}) \ e_j \] (7)

- Now arises a problem of determining the basis of \( \ker \partial^Q \): first we must determine \( D_{Q,n}^{-1} \).
- This problem is directly linked to the more general problem of determining the inverse of the elements \( \delta^*_{n-k+1} \in A(S_n) \), that to be applied here.

**Theorem**

Suppose that the parameters \( q_{ij} \)'s are in general position. Then

\[ (D_{Q,n})^{-1} = (Q_n)^{-1} \cdot E_{Q,n} \] (8)

where

\[ Q_n = (id - Q_{\{1,2\}}) \cdot (id - Q_{\{1,2,3\}}) \cdots (id - Q_{\{1,2,...,n\}}), \]

\[ E_{Q,n} = \sum_{g \in S_1^1 \times S_{n-1}} W_n(g) \cdot G, \quad W_n(g) = \prod_{i \in \text{Des}(g^{-1})} Q_{\{1,2,...,i\}} \]

with \( \text{Des}(g^{-1}) = \{ 2 \leq i \leq n-1 \mid g^{-1}(i) > g^{-1}(i+1) \} \) (the descent set of \( g^{-1} \in S_n \)).

**Theorem**

Let \( B_Q \) is generic weight subspace and the \( Q \)-cocycle condition is satisfied. Then \( X_j \in \ker \partial^Q \) is given by:

\[ X_j = (C_{Q,n} \cdot (Q_{n-1})^{-1} \cdot E_{Q,n}) \ e_j \]

with

\[ Q_{n-1} = (id - Q_{\{1,2\}}) \cdot (id - Q_{\{1,2,3\}}) \cdots (id - Q_{\{1,2,...,n-1\}}). \]
By using detailed studies (that will not be shown here), we can state the following Theorem.

**Theorem**

Let $B_Q$ is generic weight subspace and the $Q$-cocycle condition is satisfied. Let us denote $Q' = Q \setminus \{l_1, l_2\} = \{l_3, \ldots, l_n\}$. Then $\dim(\ker \partial^Q) = (n - 2)!$ and the nontrivial basic constants in the space $C_Q$ are given by

$$C_{l_1l_2j_3\ldots j_n} = (C_{Q,n} \cdot (Q_{n-1})^{-1} \cdot E_{Q,n}) \ e_{l_1l_2j_3\ldots j_n}$$

for every $j_3 \ldots j_n \in \hat{Q}'$, where $\hat{Q}'$ = the set of all distinct permutations of the set $Q'$.

In what follows we will apply the iterated $q$-commutators defined by

$$Y_{i_1\ldots i_p} = Y_{i_1\ldots i_{p-1}} e_{i_p} - q_{i_pj_1} \cdots q_{i_p i_p-1} e_{i_p} Y_{i_1\ldots i_{p-1}} \ \text{with} \ Y_{i_1} = e_{i_1}.$$
Theorem

Let $Q = \{l_1, \ldots, l_n\}$ and $Q' = Q\{ l_1, l_2 \} = \{l_3, \ldots, l_n\}$. If the $Q$-cocycle condition is satisfied (i.e. $\sigma_Q = 1$), then nontrivial basic constants in generic weight subspace $B_Q \subseteq B$ are given by:

$$C_{l_1 l_2 j_3 \ldots j_n} = ((Q_{n-1})^{-1} \cdot E_{Q,n}) \cdot Y_{l_1 l_2 j_3 \ldots j_n}$$  (9)

for every $j_3 \ldots j_n \in \widehat{Q}'$.

This Theorem is a direct consequence of the last two theorems. Recall, $\dim (\ker \partial^Q) = (n - 2)!$

Note that nontrivial basic constants given by (9) can be rewritten as

$$C_{l_1 l_2 j_3 \ldots j_n} = \sum_{g \in S_1^1 \times S_{n-1}} \left( \prod_{i \in \text{Des}(g^{-1})} Q_{\{1,2,\ldots,i\}} \right) \cdot G$$

where $g \in S_1^1 \times S_{n-1}$ fixes the first index.

The right hand side of the last formula is composed in terms of $(n - 1)!$ iterated $q$-commutators $Y_{l_1 \xi}$ such that the first index $l_1 \in Q$ is fixed and the remaining $(n - 1)$ indices $\xi = l_2 j_3 \ldots j_n$ vary.
Let us denote:

\[ x^* := \frac{1}{1 - x}, \quad x^+ := \frac{x}{1 - x} \]  \hspace{1cm} (11)

**Example**

1. If \( \sigma_{l_1 l_2} = 1 \), then in the generic weight subspace \( B_{l_1 l_2} \) there is one nontrivial basic constant, given by \( C_{l_1 l_2} = Y_{l_1 l_2} \).

2. Let \( \sigma_{l_1 l_2 l_3} = 1 \). Then in \( B_{l_1 l_2 l_3} \) there is one nontrivial basic constant:

\[
C_{l_1 l_2 l_3} = \frac{1}{1 - \sigma_{l_1 l_2}} Y_{l_1 l_2 l_3} + \frac{q_{l_3 l_2} \sigma_{l_1 l_3}}{1 - \sigma_{l_1 l_3}} Y_{l_1 l_3 l_2}
\]

which by using (11) can be rewritten as: \( C_{l_1 l_2 l_3} = \sigma_{l_1 l_2}^* Y_{l_1 l_2 l_3} + q_{l_3 l_2} \sigma_{l_1 l_3}^+ Y_{l_1 l_3 l_2} \).

3. If \( \sigma_{l_1 l_2 l_3 l_4} = 1 \), then in \( B_{l_1 l_2 l_3 l_4} \) there are two nontrivial basic constants:

\[
C_{l_1 l_2 l_3 l_4} = \sigma_{l_1 l_2}^* \sigma_{l_1 l_2 l_3}^* Y_{l_1 l_2 l_3 l_4} + q_{l_4 l_3} \sigma_{l_1 l_2}^* \sigma_{l_1 l_2 l_4}^+ Y_{l_1 l_2 l_4 l_3} + q_{l_3 l_2} \sigma_{l_1 l_3}^+ \sigma_{l_1 l_2 l_3}^* Y_{l_1 l_3 l_2 l_4} \]
\[
+ q_{l_4 l_2} q_{l_4 l_3} \sigma_{l_1 l_4}^+ \sigma_{l_1 l_2 l_4}^* Y_{l_1 l_4 l_2 l_3} + q_{l_3 l_2} q_{l_4 l_2} q_{l_4 l_3} \sigma_{l_1 l_4}^+ \sigma_{l_1 l_3 l_4}^+ Y_{l_1 l_4 l_3 l_2}
\]

\[
C_{l_1 l_2 l_4 l_3} = q_{l_3 l_4} \sigma_{l_1 l_2}^* \sigma_{l_1 l_2 l_3}^+ Y_{l_1 l_2 l_3 l_4} + q_{l_3 l_2} \sigma_{l_1 l_4}^+ \sigma_{l_1 l_2 l_4}^* Y_{l_1 l_2 l_4 l_3} \]
\[
+ q_{l_3 l_2} q_{l_3 l_4} \sigma_{l_1 l_3}^+ \sigma_{l_1 l_2 l_3}^+ Y_{l_1 l_3 l_2 l_4} + q_{l_3 l_2} q_{l_3 l_4} q_{l_4 l_2} \sigma_{l_1 l_3}^+ \sigma_{l_1 l_3 l_4}^+ Y_{l_1 l_3 l_4 l_2} \]
\[
+ q_{l_3 l_2} q_{l_4 l_2} q_{l_4 l_3} \sigma_{l_1 l_4}^+ \sigma_{l_1 l_3 l_4}^+ Y_{l_1 l_4 l_3 l_2}
\]

Here we have used the lexicographical ordering and (11).
If \( \sigma_{l_1 l_2 l_3 l_4 l_5} = 1 \), then in \( B_{l_1 l_2 l_3 l_4 l_5} \) there are six nontrivial basic constants, each consisted of 24 terms. Here we show only the first; the remaining five can be obtained from \( C_{l_1 l_2 l_3 l_4 l_5} \).

\[
C_{l_1 l_2 l_3 l_4 l_5} = \sigma_{l_1 l_2}^{*} \sigma_{l_1 l_2 l_3}^{*} \sigma_{l_1 l_2 l_3 l_4} Y_{l_1 l_2 l_3 l_4 l_5} + q_{l_4 l_3} \sigma_{l_1 l_2}^{*} \sigma_{l_1 l_2 l_3}^{+} \sigma_{l_1 l_2 l_3 l_4} Y_{l_1 l_2 l_3 l_4 l_5} + q_{l_4 l_3} \sigma_{l_1 l_2}^{+} \sigma_{l_1 l_2 l_3}^{*} \sigma_{l_1 l_2 l_3 l_4} Y_{l_1 l_2 l_3 l_4 l_5} + q_{l_4 l_3} \sigma_{l_1 l_2}^{*} \sigma_{l_1 l_2 l_3}^{*} \sigma_{l_1 l_2 l_3 l_4} Y_{l_1 l_2 l_3 l_4 l_5} + q_{l_4 l_3} \sigma_{l_1 l_2}^{*} \sigma_{l_1 l_2}^{*} \sigma_{l_1 l_2 l_3}^{+} \sigma_{l_1 l_2 l_3 l_4} Y_{l_1 l_2 l_3 l_4 l_5}
\]
THANK YOU FOR YOUR ATTENTION