ENDOSCOPIC TRANSFER FOR UNITARY GROUPS AND HOLOMORPHY OF ASAI L-FUNCTIONS

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ABSTRACT. The analytic properties of the complete Asai L-functions attached to cuspidal automorphic representations of the general linear group over a quadratic extension of a number field are obtained. The proof is based on the comparison of the Langlands-Shahidi method and Mok's endoscopic classification of automorphic representations of quasi-split unitary groups.

CONTENTS

INTRODUCTION

In this paper we study the analytic properties of the complete Asai L-function attached to a cuspidal automorphic representation of the general linear group over a quadratic extension of a

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number field. The approach is based on the Langlands-Shahidi method, combined with the knowledge of the poles of Eisenstein series coming from a recent endoscopic classification of automorphic representations of the quasi-split unitary groups by Mok [Mok].

In order to state the main result more precisely, we introduce some notation. Let E/F be a quadratic extension of number fields, and let θ be the unique non-trivial element in the Galois group $Gal(E/F)$. Let $\mathbb{A}_E$ and $\mathbb{A}_F$ be the rings of adèles of E and F, respectively. Let $\widehat{\delta}$ be any extension to $\mathbb{A}_E^\times/E^\times$ of the quadratic character of $\mathbb{A}_F^\times/F^\times$ attached to E/F by class field theory.

For a cuspidal automorphic representation σ of $GL_n(\mathbb{A}_E)$, we denote by σ^θ its Galois conjugate and by $\widetilde{\sigma}$ its contragredient representation. We say that σ is Galois self-dual if $\sigma \cong \widetilde{\sigma}^\theta$, that is, σ is isomorphic to its Galois conjugate contragredient.

Let $L(s, \sigma, r_A)$ denote the complete Asai L-function attached to σ and the Asai representation r_A via the Langlands-Shahidi method. See Sect. 2.1 for a definition. Our main result on the holomorphy and non-vanishing of the Asai L-function $L(s, \sigma, r_A)$ is the following theorem (see Theorem 4.3).

Theorem. *Let σ be a cuspidal automorphic representation of $GL_n(\mathbb{A}_E)$. Let $L(s, \sigma, r_A)$ (respectively, $L(s, \sigma \otimes \widehat{\delta}, r_A)$) be the Asai (respectively, twisted Asai) L-function attached to σ, where $\widehat{\delta}$ is any extension to $\mathbb{A}_E^\times/E^\times$ of the quadratic character of $\mathbb{A}_F^\times/F^\times$ attached to the extension E/F by class field theory.*

(1) *If σ is* not *Galois self-dual, i.e., $\sigma \not\cong \widetilde{\sigma}^\theta$, then $L(s, \sigma, r_A)$ is entire. It is non-zero for $Re(s) \geq 1$ and $Re(s) \leq 0$.*
(2) *If σ is Galois self-dual, i.e., $\sigma \cong \widetilde{\sigma}^\theta$, then*
    (a) *$L(s, \sigma, r_A)$ is entire, except for possible simple poles at $s = 0$ and $s = 1$, and non-zero for $Re(s) \geq 1$ and $Re(s) \leq 0$;*
    (b) *exactly one of the L-functions $L(s, \sigma, r_A)$ and $L(s, \sigma \otimes \widehat{\delta}, r_A)$ has simple poles at $s = 0$ and $s = 1$, while the other is holomorphic at those points.*

The idea of the proof is to consider the Eisenstein series attached to σ on the quasi-split unitary group $U_{2n}(\mathbb{A}_F)$ defined by the quadratic extension E/F, where σ is viewed as a representation of the Levi factor of the Siegel maximal parabolic subgroup of U_{2n} in $2n$ variables. We look at the contribution of this Eisenstein series to the residual spectrum from two different points of view. On the one hand, by the Langlands-Shahidi method [Sha10], the poles of the Eisenstien series for the complex argument in the positive Weyl chamber are determined by certain ratio of the Asai L-functions. The residues at such a pole span a residual representation of $U_{2n}(\mathbb{A}_F)$. On the other hand, this residual representation should have an Arthur parameter, according to Mok's endoscopic classification of automorphic representations of quasi-split unitary groups [Mok] (see also [Art13], [Art05]). Comparing the possible Arthur parameters and poles of Asai L-functions, we are able to deduce the analytic properties of these L-functions.

Mok's work, as well as Arthur's, still depends on the stabilization of the twisted trace formula for the general linear group. Hence, our result is also conditional on this stabilization. This issue is being considered by Waldspurger [WalA], [WalB], [WalC]. In the paper, we always make a remark when a partial result could have been obtained without using Mok's work. In fact, the crucial insight coming from endoscopic classification is holomorphy of the Asai L-function $L(s, \sigma, r_A)$ inside the critical strip $0 < Re(s) < 1$.

In a previous work [Grb11], the first named author applied this method to the complete exterior and symmetric square L-functions attached to a cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$. It relies on Arthur's endoscopic classification of automorphic representations for split classical groups [Art13], [Art05]. The result and the approach are of the same nature as the above theorem for Asai L-functions. The approach has also already been used in [JLZ13] as a part of a long term project to study endoscopy via descent by Jiang, Liu and Zhang.

A different approach to describing the analytic properties of L-functions is that of integral representations. However, this approach usually gives holomorphy of *partial* L-functions, which is weaker than our result due to ramification or problems at archimedean places. For the exterior square L-function this was pursued in [BF90], [KR12], [Bel12], for the symmetric square L-function in [BG92] and more generally for twisted symmetric square in a series of papers [Tak14], [TakA], [TakB], and for the Asai L-functions in [Fli88], [FZ95], [AR05].

The paper is organized as follows. In Section 1 we introduce the unitary group structure and fix the notation. In Section 2 the relation between poles of Eisenstein series on quasi-split unitary groups and the Asai L-functions is investigated. Section 3 provides a definition of Arthur parameters and packets for quasi-split unitary groups in terms of results of Mok. Finally, in Section 4 we prove the main result on the analytic properties of Asai L-functions.

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1. The quasi-split unitary groups

1.1. **Definition and basic structure.** Let E/F be a quadratic extension of number fields. The non-trivial Galois automorphism in the Galois group $Gal(E/F)$ is denoted by θ. Let $N_{E/F}$ denote the norm map from E to F. Let $\mathbb{A}_F$ and $\mathbb{A}_E$ be the rings of adèles of F and E, respectively, and $\mathbb{A}_F^\times$ and $\mathbb{A}_E^\times$ the corresponding groups of idèles.

The quadratic character of $\mathbb{A}_F^\times/F^\times$ attached to E/F by class field theory is denoted by $\delta_{E/F}$. We always identify $\delta_{E/F}$ with the corresponding character of the Weil group W_F of F under class field theory. Let $\widehat{\delta}$ be any extension of $\delta_{E/F}$ to a character of $\mathbb{A}_E^\times/E^\times$. Such extension is not unique.

We denote by F_v the completion of F at the place v. If a place v of F does not split in E, we always denote by w the unique place of E lying over v. Then, E_w/F_v is a quadratic extension of local fields. If v splits in E, we denote by w_1 and w_2 the two places of E lying over v. Then, we have $E_{w_1} \cong E_{w_2} \cong F_v$. We use F_∞ to denote the product of F_v over archimedean places.

For an integer $N \geq 2$, we consider in this paper the F-quasi-split unitary group U_N in N variables defined by the extension E/F, viewed as an algebraic group over F. More precisely, U_N is a group scheme over F, whose functor of points is defined as follows. Consider θ as an element of the Galois group $Gal(\overline{F}/F)$ trivial on $\overline{F}/E$, where $\overline{F}$ is a fixed algebraic closure of F. Let V be an N-dimensional vector space over E. We fix a form on V as in [KK04] and [KK05], that is, let

$$J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \quad \text{and} \quad J'_N = \begin{cases} \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}, & \text{for } N = 2n, \\ \begin{pmatrix} 0 & 0 & J_n \\ 0 & 1 & 0 \\ -J_n & 0 & 0 \end{pmatrix}, & \text{for } N = 2n+1. \end{cases}$$

Then, the functor of points of U_N is given by

$$U_N(R) = \left\{ g \in GL_{E\otimes_F R}(V \otimes_F R) \, : \, {}^*g J'_N g = J'_N \right\}$$

for any F-algebra R, where ${}^*g = {}^t g^\theta$ is the conjugate transpose of g. In particular, the F-points of U_N are given as

$$U_N(F) = \left\{ g \in GL_N(E) \, : \, {}^*g J'_N g = J'_N \right\}.$$

Writing $N = 2n$ if N is even, and $N = 2n+1$ if N is odd, the F-rank of U_N in both cases equals $n \geq 1$. For $N = 1$, the unitary group U_1 in one variable is obtained by inserting $N = 1$ in the definition of U_N. Its F-points are nothing else than

$$U_1(F) = \left\{ x \in E^\times \, : \, \theta(x)x = 1 \right\},$$

which is the norm one subgroup E^1 of $E^\times$.

For $m \geq 1$, let $G_m = Res_{E/F} GL_m$ be the algebraic group over F obtained from the general linear group GL_m over E by restriction of scalars from E to F. If $m \leq n$, it appears in the Levi factors of parabolic subgroups of U_N.

We fix the Borel subgroup P_0 of U_N consisting of upper-triangular matrices. Let $P_0 = M_0 N_0$, where M_0 is a maximally split maximal torus of U_N, i.e., containing a maximal split torus of U_N (see [Sha10, Chapter I]), and N_0 the unipotent radical of P_0. Then,

$$M_0 \cong \begin{cases} G_1 \times \cdots \times G_1, & \text{for } N = 2n, \\ G_1 \times \cdots \times G_1 \times U_1, & \text{for } N = 2n+1, \end{cases}$$

with n copies of G_1, so that the F-points of M_0 are given by

$$M_0(F) = \begin{cases} \left\{ \mathrm{diag}(t_1, \ldots, t_n, \theta(t_n)^{-1}, \ldots, \theta(t_1)^{-1}) \, : \, t_i \in E^\times \right\}, & \text{for } N = 2n, \\ \left\{ \mathrm{diag}(t_1, \ldots, t_n, t, \theta(t_n)^{-1}, \ldots, \theta(t_1)^{-1}) \, : \, t_i \in E^\times, t \in E^1 \right\} & \text{for } N = 2n+1. \end{cases}$$

Let A_0 be a maximal F-split torus of U_N, which is a subtorus of M_0. Then,

$$A_0(F) = \begin{cases} \left\{ \mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \, : \, t_i \in F^\times \right\}, & \text{for } N = 2n, \\ \left\{ \mathrm{diag}(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1}) \, : \, t_i \in F^\times \right\}, & \text{for } N = 2n+1. \end{cases}$$

The absolute root system $\Phi = \Phi(U_N, M_0)$ of U_N with respect to M_0 is of type A_{N-1}. The root system $\Phi_{red} = \Phi(U_N, A_0)$ of U_N with respect to A_0 is a reduced root system. It is of type C_n for $N = 2n$ and of type BC_n for $N = 2n+1$. We make the choice of positive roots according to the fixed Borel subgroup P_0, and let Δ be the set of simple roots. We order the simple roots as in Bourbaki [Bou68].

Let P be the Siegel maximal proper standard parabolic F-subgroup of U_N. That is, it is defined, in a standard fashion, by a subset of simple roots obtained by removing the last simple root in the Bourbaki ordering (cf. [Bou68] and [Sha10, Sect. 1.2]). Let $P = M_P N_P$ be the Levi decomposition of P, where

$$M_P \cong \begin{cases} G_n, & \text{for } N = 2n, \\ G_n \times U_1, & \text{for } N = 2n+1, \end{cases}$$

is the Levi factor, and N_P the unipotent radical.

1.2. ***L*-groups.** The L-group of U_N is a semidirect product

$$^{L}U_N = GL_N(\mathbb{C}) \rtimes W_F,$$

where W_F is the Weil group of F. It is acting on the connected component $^{L}U_N^{\circ} = GL_N(\mathbb{C})$ through the quotient $W_F/W_E \cong Gal(E/F)$. The action of the non-trivial Galois automorphism $\theta \in Gal(E/F)$ is given by

$$\theta(g) = J_N'^{-1}{}^t g^{-1} J_N'$$

for all $g \in GL_N(\mathbb{C})$.

The L-group of the Levi factor M_P is a semidirect product

$$^{L}M_P = \begin{cases} GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) \rtimes W_F, & \text{for } N = 2n, \\ GL_n(\mathbb{C}) \times GL_1(\mathbb{C}) \times GL_n(\mathbb{C}) \rtimes W_F, & \text{for } N = 2n+1, \end{cases}$$

where the Weil group W_F acts through the quotient $W_F/W_E \cong Gal(E/F)$ on the connected component of the L-group, and $\theta \in Gal(E/F)$ acts by interchanging the two $GL_n(\mathbb{C})$ factors.

2. Eisenstein series and Asai L-functions

In this section we relate the analytic behavior of the Eisenstein series on the unitary group supported in the Siegel parabolic subgroup to a ratio of the Asai L-functions appearing in its constant term. For the study of analytic properties of the Asai L-functions, it is sufficient to consider the even quasi-split unitary group U_{2n}. However, for completeness and future reference, we also study the Eisenstein series in the odd case.

We retain all the notation of Section 1. So, P is the Siegel maximal proper standard parabolic F-subgroup of U_N, with the Levi factor $M_P \cong G_n$ if $N = 2n$ is even, and $M_P \cong G_n \times U_1$ if $N = 2n+1$ is odd, and the unipotent radical N_P. Recall that $G_n = Res_{E/F} GL_n$.

2.1. **Asai L-functions.** Let σ be a cuspidal automorphic representation of $G_n(\mathbb{A}_F) \cong GL_n(\mathbb{A}_E)$ and ν a character of $U_1(\mathbb{A}_F) \cong \mathbb{A}_E^1$ trivial on $U_1(F) \cong E^1$. To make a convenient normalization in the case of odd unitary group, as in [Rog90] and [Gol94, Sect. 6], we denote by $\widehat{\nu}$ a unitary character of $GL_n(\mathbb{A}_E)$ given by

$$\widehat{\nu}(g) = \nu(\det(g^* g^{-1}))$$

for all $g \in GL_n(\mathbb{A}_E)$. Observe that $\det(g^* g^{-1})$ is of norm one. Then we define a cuspidal automorphic representation Σ of the Levi factor $M_P(\mathbb{A}_F)$ as

$$\Sigma = \begin{cases} \sigma, & \text{for } N = 2n, \\ (\sigma\widehat{\nu}) \otimes \nu, & \text{for } N = 2n+1. \end{cases}$$

More precisely, in the case of odd unitary group the action of Σ is given by

$$\Sigma(g,t) = \sigma(g)\nu(\det(g^* g^{-1}))\nu(t)$$

for $g \in GL_n(\mathbb{A}_E)$ and $t \in \mathbb{A}_E^1$. We always assume that Σ is irreducible unitary and trivial on $A_P(F_\infty)^\circ$, the identity connected component of $A_P(F_\infty)$, where A_P is a maximal F-split torus in the center of M_P. The last condition is not restrictive. It is just a convenient normalization, obtained by twisting by a unitary character, which makes the poles of Eisenstein series real.

We define first the local L-functions. Let v be a place of F. By extension of scalars from F to F_v, we may view the unitary group U_N as an algebraic group over F_v. This algebraic group is denoted by $U_{N,v}$. Then we have the parabolic subgroup P_v of $U_{N,v}$ defined over F_v with Levi decomposition $M_{P,v}N_{P,v}$, where $M_{P,v}$ is the Levi factor and $N_{P,v}$ the unipotent radical.

In the case of the even unitary group, i.e., $N = 2n$, the adjoint representation r_v of the L-group ${}^L M_{P,v}$ on the Lie algebra ${}^L\mathfrak{n}_{P,v}$ of the L-group of $N_{P,v}$ is irreducible for all places v of F. If v does not split in E, then r_v is called the Asai representation, as it generalizes the case considered by Asai in [Asa77]. We denote it by $r_{A,v}$. This situation is labeled ${}^2A_{2n-1} - 2$ in the list of [Sha88, Sect. 4] and [Sha10, App. C]. Explicit action of $r_{A,v}$ is given in [Gol94, Sect. 3].

In the case of the odd unitary group, i.e., $N = 2n+1$, the analogous adjoint representation is a direct sum $r_{1,v} \oplus r_{2,v}$ of two irreducible representations for all places v of F, ordered as in [Sha90]. If v does not split in E, then $r_{2,v}$ is the twisted Asai representation $r_{A,v} \otimes \delta_{E_w/F_v}$, where w is the unique place of E lying over v. This situation is labeled ${}^2A_{2n} - 3$ in the list of [Sha88, Sect. 4] and [Sha10, App. C].

For a cuspidal automorphic representation Σ of $M_P(\mathbb{A}_F)$, let $\Sigma \cong \otimes'_v \Sigma_v$ be a decomposition into a restricted tensor product over all places. Let R_v be one of the adjoint representations defined above. Then the local L-functions $L(s, \Sigma_v, R_v)$ attached to Σ_v and R_v are defined as follows.

- at archimedean places the Artin L-functions attached to the Langlands parameter of Σ_v as in [Sha85] (see also [Sha10, Sect. 8.2], and [Lan89] where the Langlands parametrization over reals was first introduced),
- at unramified non-archimedean places given in terms of Satake parameters of Σ_v (cf. [Sha88], [Sha10, Def. 2.3.5], and also [HLR86] where Asai's name came up first),
- at the remaining non-archimedean places defined using the Langlands-Shahidi method [Sha90, Sect. 7] (see also [Sha10, Sect. 8.4]).

The corresponding global L-functions are defined as the analytic continuation from the domain of convergence of the product over all places of local L-functions $L(s, \Sigma_v, R_v)$. According to [Lan71], see also [Sha10, Sect. 2.5], the product over all places defining the global L-functions converges absolutely in some right half-plane $Re(s) > C$, where C is sufficiently large.

The global L-function obtained in this way from $\Sigma = \sigma \cong \otimes'_v \sigma_v$ and $R_v = r_{1,v}$ is denoted by $L(s, \sigma, r_A)$ and called the Asai L-function attached to σ. Its analytic properties are the main concern of this paper.

The global L-function obtained from $\Sigma \cong \otimes'_v \Sigma_v$ and $R_v = r_{2,v}$ is denoted by $L(s, \Sigma, r_A \otimes \delta_{E/F})$ and called the twisted Asai L-function attached to Σ. In fact, it is the same as the Asai L-function $L(s, \sigma \otimes \widehat{\delta}, r_A)$ attached to $\sigma \otimes \widehat{\delta}$ (see [Gol94]). Hence, the analytic properties of the twisted Asai L-function follow from the analytic properties of the Asai L-function attached to a twisted representation. Recall that $\widehat{\delta}$ is any extension of the quadratic character $\delta_{E/F}$ to $\mathbb{A}_E^\times$.

Finally, as shown in [Gol94], the choice of the normalization of Σ in the case of odd unitary groups implies that the global L-function obtained from $\Sigma \cong \otimes'_v \Sigma_v$ and $R_v = r_{1,v}$ is the same as the principal L-function $L(s, \sigma)$ attached to σ by Godement-Jacquet [GJ72]. Its analytic properties are well known. It is entire, unless $n = 1$ and σ is the trivial character $\mathbf{1}_{\mathbb{A}_E^\times}$ of $\mathbb{A}_E^\times$. In that case $L(s, \mathbf{1}_{\mathbb{A}_E^\times})$ is holomorphic except for simple poles at $s = 0$ and $s = 1$.

2.2. **Eisenstein series.** For $s \in \mathbb{C}$ and Σ a cuspidal automorphic representation of $M_P(\mathbb{A}_F)$ as above, let

$$I(s, \Sigma) = \begin{cases} \mathrm{Ind}_{P(\mathbb{A}_F)}^{U_N(\mathbb{A}_F)} \left(\sigma |\det|_E^s\right), & \text{for } N = 2n, \\ \mathrm{Ind}_{P(\mathbb{A}_F)}^{U_N(\mathbb{A}_F)} \left(\sigma\widehat{\nu} |\det|_E^s \otimes \nu\right), & \text{for } N = 2n+1, \end{cases}$$

be the induced representation, where the induction is normalized. As in [Sha10, page 108], we realize $I(s,\Sigma)$ for all $s \in \mathbb{C}$ on the same space W_Σ of smooth functions

$$f : N_P(\mathbb{A}_F)M_P(F)A_P(F_\infty)^\circ \backslash U_N(\mathbb{A}_F) \to \mathbb{C},$$

K-finite with respect to a fixed maximal compact subgroup K of $U_N(\mathbb{A}_F)$ compatible to P (as in [MW95, Sect. I.1.4]), and such that the function on $M_P(\mathbb{A}_F)$ given by the assignment $m \mapsto f(mg)$ for $m \in M_P(\mathbb{A}_F)$ belongs to the space of Σ for all $g \in U_N(\mathbb{A}_F)$. The dependence on $s \in \mathbb{C}$ is hidden in the action of $U_N(\mathbb{A}_F)$.

Given $f \in W_\Sigma$ and $s \in \mathbb{C}$, set

$$f_s(g) = f(g)\exp\langle s + \rho_P, H_P(g)\rangle$$

for all $g \in U_N(\mathbb{A}_F)$. Here ρ_P is the half-sum of positive roots not being the roots of M_P, and H_P is a map

$$H_P : U_N(\mathbb{A}_F) \to \mathrm{Hom}(X(M_P)_F, \mathbb{R}),$$

where $X(M_P)_F$ denotes the group of F-rational characters of M_P, defined on $m = (m_v)_v \in M_P(\mathbb{A}_F)$ by the condition

$$\exp\langle \chi, H_P(m)\rangle = \prod_v |\chi(m_v)|_v$$

for every $\chi \in X(M_P)_F$, and extended via Iwasawa decomposition to $U_N(\mathbb{A}_F)$ trivially on the unipotent radical $N_P(\mathbb{A}_F)$ and the fixed maximal compact subgroup K (cf. [Sha10, Sect. 1.3]). Then, the Eisenstein series is defined as the analytic continuation from the domain of convergence $Re(s) > \rho_P$ of the series

$$E(f,s)(g) = \sum_{\gamma \in P(F)\backslash U_N(F)} f(\gamma g)\exp\langle s+\rho_P, H_P(\gamma g)\rangle = \sum_{\gamma \in P(F)\backslash U_N(F)} f_s(\gamma g)$$

for $g \in U_N(\mathbb{A}_F)$. The Eisenstein series $E(f,s)$ has a finite number of simple poles in the real interval $0 < s \le \rho_P$, and all other poles have $Re(s) < 0$ (cf. [MW95, Sect. IV.1.11 and IV.3.12]). The residue of the Eisenstein series $E(f,s)$ at $s > 0$ is a square-integrable automorphic form on $U_N(\mathbb{A}_F)$, but not cuspidal, thus belonging to the residual spectrum of $U_N(\mathbb{A}_F)$. In fact, such residues for all $f \in W_\Sigma$ span the summand of the residual spectrum of $U_N(\mathbb{A}_F)$ with cuspidal support in Σ (see [MW95, Sect. III.2.6] or [FS98, Sect. 1] for the decomposition of the space of automorphic forms with respect to their cuspidal support).

2.3. **Asai L-functions in the constant term.** Now we prove that the poles of Eisenstein series $E(f,s)(g)$ for $Re(s) > 0$ coincide with the poles for $Re(s) > 0$ of the ratio of L-functions appearing in its constant term.

Theorem 2.1. *Let σ be a cuspidal automorphic representation of $G_n(\mathbb{A}_F) \cong GL_n(\mathbb{A}_E)$ and ν a unitary character of $U_1(\mathbb{A}_F) \cong \mathbb{A}_E^1$ trivial on $U_1(F) \cong E^1$. As in Sect. 2.2, form a cuspidal automorphic representation Σ of the Levi factor $M_P(\mathbb{A}_F)$ in U_N. Then, the poles with $Re(s) > 0$ of the Eisentein series $E(f,s)$ for some $f \in W_\Sigma$ coincide with the poles with $Re(s) > 0$ of*

$$\begin{cases} \frac{L(2s,\sigma,r_A)}{L(1+2s,\sigma,r_A)}, & \text{if } N = 2n, \\ \frac{L(s,\sigma)}{L(1+s,\sigma)} \cdot \frac{L(2s,\sigma\otimes\widehat{\delta},r_A)}{L(1+2s,\sigma\otimes\widehat{\delta},r_A)}, & \text{if } N = 2n+1, \end{cases}$$

where $\widehat{\delta}$ is any extension to $\mathbb{A}_E^\times/E^\times$ of the quadratic character $\delta_{E/F}$ of $\mathbb{A}_F^\times/F^\times$ attached to E/F by class field theory.

Remark 2.2. Observe the factor two appearing in the argument $2s$ of the Asai L-function in the case of even unitary groups. The reason is that we have chosen, as in [Sha92], the determinant character to normalize the identification with $\mathbb{C}$ of the complex parameter s in the Eisenstein series, instead of the character $\tilde{\alpha}$ given in terms of the half-sum of positive roots and the coroot of the unique simple root α not being a root of M_P, as in [Sha90].

Proof of Theorem 2.1. This is an application of the Langlands spectral theory, using the Langlands-Shahidi method to normalize the intertwining operator.

The poles of Eisenstein series $E(f,s)$ coincide with the poles of its constant term $E(f,s)_P$ along P. The constant term is defined as

$$E(f,s)_P(g) = \int_{N_P(F)\backslash N_P(\mathbb{A}_F)} E(f,s)(ng)dn,$$

where dn is a fixed Haar measure on $N_P(\mathbb{A}_F)$. On the other hand, the constant term can be written as

$$E(f,s)_P(g) = f_s(g) + \big(M(s,\Sigma,w_0)f\big)_{-s}(g),$$

where $M(s,\Sigma,w_0)$ is the standard intertwining operator. Here w_0 is the unique non-trivial Weyl group element such that $w_0(\alpha)$ is a simple root for all simple roots α except the last one in the ordering of [Bou68].

As in [Sha10, page 109], the standard intertwining operator is defined as the analytic continuation from the domain of convergence of the integral

$$M(s,\Sigma,w_0)f(g) = \left(\int_{N_P(\mathbb{A}_F)} f_s(\dot{w}_0^{-1}ng)dn\right)\exp\langle s-\rho_P, H_P(g)\rangle,$$

where $\dot{w}_0$ is a fixed representative for w_0 in $U_N(F)$. For $s\in\mathbb{C}$ away from poles, the assignment $f\mapsto M(s,\Sigma,w_0)f$ defines a linear map on W_Σ, which depends on s. It intertwines the actions of $I(s,\Sigma)$ and $I(-s,\Sigma^{w_0})$. Let σ^θ denote σ conjugated by the non-trivial Galois automorphism $\theta\in Gal(E/F)$, that is, $\sigma^\theta(m)=\sigma(m^\theta)$ for all $m\in GL_n(\mathbb{A}_E)$. Note that in our case the conjugation by w_0 amounts to taking $\widetilde{\sigma}^\theta$, where $\widetilde{\sigma}$ is the contragredient of σ. In the case of odd unitary groups this means that $\Sigma^{w_0}\cong\widetilde{\sigma}^\theta\widehat{\nu}\otimes\nu$ (see [Gol94]).

It is clear from the expression for the constant term that the poles of the Eisenstein series are the same as those of the standard intertwining operator. We apply the Langlands-Shahidi method to normalize this operator. The normalizing factor in this situation, labeled ${}^2A_{2n-1}-2$ for the even unitary group and ${}^2A_{2n}-3$ for the odd unitary group in the list of [Sha88, Sect. 4] and [Sha10, App. C], is given in terms of L-functions and corresponding ε-factors as

$$r(s,\Sigma,w_0) = \begin{cases} \frac{L(2s,\sigma,r_A)}{L(1+2s,\sigma,r_A)\varepsilon(2s,\sigma,r_A)}, & \text{for } N=2n,\\ \frac{L(s,\sigma)}{L(1+s,\sigma)\varepsilon(s,\sigma)}\cdot\frac{L(2s,\sigma\otimes\widehat{\delta},r_A)}{L(1+2s,\sigma\otimes\widehat{\delta},r_A)\varepsilon(2s,\sigma\otimes\widehat{\delta},r_A)}, & \text{for } N=2n+1.\end{cases}$$

The normalized intertwining operator

$$r(s,\Sigma,w_0)^{-1}M(s,\Sigma,w_0)$$

is holomorphic and *not* identically vanishing on $I(s,\Sigma)$ for $Re(s)>0$. This is essentially a local fact proved in Lemma 2.3 below.

Assuming this fact, we now finish the proof. The holomorphy and non-vanishing of the normalized operator implies that the poles of $M(s,\Sigma,w_0)$ for $Re(s)>0$ coincide with those of $r(s,\Sigma,w_0)$. Since the ε-factors are entire and non-vanishing for all $s\in\mathbb{C}$, these are the same as the poles of the ratios of L-functions given in the theorem. □

2.4. **Holomorphy and non-vanishing of normalized intertwining operators.** It remains to show the fact that $r(s,\Sigma,w_0)^{-1}M(s,\Sigma,w_0)$ is holomorphic and non-vanishing for $Re(s)>0$. The notation is as in the proof of the previous theorem. This is essentially a local problem, because one can decompose over the places of F the action of the standard intertwining operator acting on a decomposable function using the fact that all ingredients are unramified at all but finitely many places. Hence, the problem reduces to a finite number of ramified and archimedean places, which is solved for each place separately.

We introduce some local notation first. Let $\Sigma\cong\otimes'_v\Sigma_v$ be the decomposition of Σ into a restricted tensor product, where in the case of odd unitary group $\Sigma_v=\sigma_v\widehat{\nu}_v\otimes\nu_v$. We denote the local standard intertwining operator by $M(s,\Sigma_v,w_0)$. It is defined as the analytic continuation of the local analogue of the integral defining the global operator $M(s,\Sigma,w_0)$ (see the proof of Theorem 2.1). Let $r(s,\Sigma_v,w_0)$ be the local factor at v of $r(s,\Sigma,w_0)$. We show in the lemma below that the normalized local intertwining operator

$$N(s,\Sigma_v,w_0)=r(s,\Sigma_v,w_0)^{-1}M(s,\Sigma_v,w_0)$$

is holomorphic and not identically vanishing on the local induced representation $I(s,\Sigma_v)$ for $Re(s)>0$.

Lemma 2.3. *Let Σ_v be a local component of a cuspidal automorphic representation Σ of the Levi factor $M_P(\mathbb{A}_F)$ in the unitary group U_N. Then, for $Re(s)>0$, the normalized local intertwining operator $N(s,\Sigma_v,w_0)$ is holomorphic and not identically vanishing on the induced representation $I(s,\Sigma_v)$.*

Proof. Consider first the case in which the place v of F splits in E. Then $U_N(F_v)$ is isomorphic to $GL_N(F_v)$, and the Levi factor

$$M_P(F_v)\cong\begin{cases} GL_n(F_v)\times GL_n(F_v), & \text{for } N=2n,\\ GL_n(F_v)\times GL_1(F_v)\times GL_n(F_v), & \text{for } N=2n+1.\end{cases}$$

Hence, the normalized operator considered in the lemma is attached to a unitary representation of a Levi factor $M_P(F_v)$ in $GL_N(F_v)$. The holomorphy and non-vanishing for $Re(s)>0$ follow from [MW89, Prop. I.10].

We consider now the case in which the place v of F does not split in E, and denote by w the unique place of E lying over v. Then E_w/F_v is a quadratic extension of local fields, and $U_N(F_v)$ is the quasi-split unitary group in N variables given by the extension E_w/F_v. The Levi factor $M_P(F_v)$ is isomorphic to

$$M_P(F_v)\cong\begin{cases} G_n(F_v)\cong GL_n(E_w), & \text{for } N=2n,\\ G_n(F_v)\times U_1(F_v)\cong GL_n(E_w)\times E_w^1, & \text{for } N=2n+1,\end{cases}$$

so that

$$\Sigma_v\cong\begin{cases} \sigma_w, & \text{for } N=2n,\\ (\sigma_w\widehat{\nu}_w)\otimes\nu_w, & \text{for } N=2n+1,\end{cases}$$

where σ_w is the local component of a cuspidal automorphic representation σ of $GL_n(\mathbb{A}_E)$ at the place w of E, and ν_w the local component of a unitary character ν of $\mathbb{A}_E^1$ trivial on E^1.

In particular, σ_w is unitary and generic, since it is a local component of a cuspidal automorphic representation of $GL_n(\mathbb{A}_E)$. Hence, by [Tad86] in nonarchimedean, and [Vog86] in archimedean case, there is

- a standard parabolic subgroup Q of GL_n such that the Levi factor M_Q of Q is isomorphic to $GL_{d_1} \times \cdots \times GL_{d_\ell}$, where $d_1 + \cdots + d_\ell = n$,
- unitary square-integrable representations δ_i of $GL_{d_i}(E_w)$, for $i = 1, \ldots, \ell$, and
- real numbers α_i with $0 \leq |\alpha_i| < 1/2$, for $i = 1, \ldots, \ell$,

such that σ_w is isomorphic to the fully induced representation

$$\sigma_w \cong \mathrm{Ind}_{Q(E_w)}^{GL_n(E_w)} \Big(\delta_1 |\det|^{\alpha_1} \otimes \cdots \otimes \delta_\ell |\det|^{\alpha_\ell} \Big).$$

Let R be the standard parabolic F-subgroup of U_N with the Levi factor

$$M_R \cong \begin{cases} G_{d_1} \times \cdots \times G_{d_\ell}, & \text{for } N = 2n, \\ G_{d_1} \times \cdots \times G_{d_\ell} \times U_1, & \text{for } N = 2n+1, \end{cases}$$

so that $R \subset P$ and $M_R(F_v) = M_Q(E_w)$ for $N = 2n$ and $M_R(F_v) = M_Q(E_w) \times E_w^1$ for $N = 2n+1$. Let

$$\delta = \begin{cases} \delta_1 \otimes \cdots \otimes \delta_\ell, & \text{for } N = 2n, \\ \delta_1 \widehat{\nu}_1 \otimes \cdots \otimes \delta_\ell \widehat{\nu}_\ell \otimes \nu, & \text{for } N = 2n+1, \end{cases}$$

be a unitary square-integrable representation of $M_R(F_v)$, where $\widehat{\nu}_i$ is the character of $GL_{d_i}(E_w)$ given by $\widehat{\nu}_i(h_i) = \nu(\det(h_i {}^* h_i^{-1}))$ for $h_i \in GL_{d_i}(E_w)$.

By induction in stages, the intertwining operator $N(s, \Sigma_v, w_0)$ coincides with the intertwining operator

$$N\big((s+\alpha_1, \ldots, s+\alpha_\ell), \delta, w_0\big)$$

acting on the induced representation

$$\begin{cases} \mathrm{Ind}_{R(F_v)}^{U_N(F_v)} \Big(\delta_1 |\det|^{s+\alpha_1} \otimes \cdots \otimes \delta_\ell |\det|^{s+\alpha_\ell} \Big), & \text{for } N = 2n, \\ \mathrm{Ind}_{R(F_v)}^{U_N(F_v)} \Big(\delta_1 \widehat{\nu}_1 |\det|^{s+\alpha_1} \otimes \cdots \otimes \delta_\ell \widehat{\nu}_\ell |\det|^{s+\alpha_\ell} \otimes \nu \Big), & \text{for } N = 2n+1. \end{cases}$$

By Zhang's lemma [Zha97] (see also [Kim00, Lemma 1.7]), the holomorphy of this last operator at s implies non-vanishing. Hence, to show the lemma, it is sufficient to prove the holomorphy for $Re(s) > 0$.

To prove the holomorphy for $Re(s) > 0$, we decompose the intertwining operator into a product of intertwining operators as in [Sha81, Sect. 2.1]. If we show that each factor is holomorphic for $Re(s) > 0$, then the product is holomorphic for $Re(s) > 0$ as well, and the lemma is proved. The factors are normalized intertwining operators that can be viewed as intertwining operators on representations induced from appropriate maximal proper parabolic subgroups in certain reductive groups. In our case these rank-one factors are normalized operators

$$N(2s + \alpha_i + \alpha_j, \delta_i \otimes \widetilde{\delta}_j^\theta),$$

for $1 \leq i < j \leq \ell$, acting on the induced representations

$$\mathrm{Ind}_{Q_{i,j}(E_w)}^{GL_{d_i+d_j}(E_w)} \left(\delta_i |\det|^{s+\alpha_i} \otimes \widetilde{\delta}_j^\theta |\det|^{-s-\alpha_j} \right),$$

where $Q_{i,j}$ is the maximal standard proper parabolic subgroup of $GL_{d_i+d_j}$ with the Levi factor $GL_{d_i} \times GL_{d_j}$, and normalized operators

$$\begin{cases} N(s+\alpha_k, \delta_k), & \text{for } N = 2n, \\ N(s+\alpha_k, (\delta_k \widehat{\nu}_k) \otimes \nu), & \text{for } N = 2n+1, \end{cases}$$

for $1 \leq k \leq \ell$, acting on the induced representation

$$\begin{cases} \mathrm{Ind}_{Q_k(F_v)}^{U_{2d_k}(F_v)} \left(\delta_k |\det|^{s+\alpha_k} \right), & \text{for } N = 2n, \\ \mathrm{Ind}_{Q_k(F_v)}^{U_{2d_k+1}(F_v)} \left(\delta_k \widehat{\nu}_k |\det|^{s+\alpha_k} \otimes \nu \right), & \text{for } N = 2n+1, \end{cases}$$

where Q_k is the maximal standard proper parabolic subgroup of U_{2d_k} with the Levi factor G_{d_k} if $N = 2n$, and of U_{2d_k+1} with the Levi factor $G_{d_k} \times U_1$ if $N = 2n+1$. We suppress the Weyl group element from the notation for these intertwining operators, because they are always determined by the maximal parabolic subgroup in question.

According to [Zha97, Sect. 2], the rank-one normalized intertwining operator is holomorphic for real part of its complex parameter greater than the first negative point of reducibility of the induced representation on which it acts. For $Re(s) > 0$, using the bound on α_i, we have

$$Re(s + \alpha_i + \alpha_j) > -1 \quad \text{and} \quad Re(s + \alpha_k) > -1/2.$$

But these two bounds are precisely the first negative points of reducibility in the cases $Q_{i,j} \subset GL_{d_i+d_j}$ and $Q_k \subset U_{2d_k}$ or U_{2d_k+1}. This essentially follows from the standard module conjecture, proved in [Vog78] for any quasi-split real group, and thus for complex groups as well, and in [Mui01] for quasi-split classical groups over a p-adic field. In [CS98, Sect. 5] the reducibility points are determined in terms of local coefficients over any local field. A convenient reference making explicit the first reducibility points of such complementary series using local coefficients for any quasi-split classical group over a local field of characteristic zero is [LMT04, Lemma 2.6 and 2.7]. For the general linear group the reducibility is obtained in [Zel80] over a p-adic field, in [Spe82] over reals, and in [Wal79] over complex numbers (see also [Kim00, Lemma 2.10]). For the unitary group over a non-archimedean field it is obtained in [Gol94, Sect. 3 and 6], applying the general reducibility result of [Sha90], while at an archimedean place, the L-functions in the local coefficient that control reducibility are the L-functions of the restriction to $\mathbb{R}^\times$ of a character of $\mathbb{C}^\times$ (see [LMT04, Lemma 2.6]). Thus, the rank-one factors are all holomorphic and the lemma is proved. □

Remark 2.4. Kim and Krishnamurty have proved in [KK04] and [KK05] the holomorphy and non-vanishing of normalized intertwining operators for a representation of the Levi factor of any maximal proper parabolic subgroup of U_N, which is a local component of a generic cuspidal automorphic representation. Since in our case all cuspidal automorphic representations of the Levi factor are generic, Lemma 2.3 follows from their work. Their proof uses their stable base change lift and bounds towards Ramanujan conjecture obtained by Luo-Rudnick-Sarnak [LRS99] to bound the exponents on the unitary group. In our case these bounds are not required because our unitary factor in the Levi is either trivial or rank zero. This simplifies the proof.

3. Arthur parameters for unitary groups

Our next task is to introduce the notion of Arthur parameters and the endoscopic classification of automorphic representations for the quasi-split unitary group U_N in N variables. We consider both even and odd case for completeness, although for the application to the analytic properties of the Asai L-functions only the even case is required.

In a recent preprint of Mok [Mok], Arthur's work [Art13] (see also [Art05]) is extended to the case of quasi-split unitary groups. As in Arthur's paper [Art04], we avoid the conjectural Langlands group by describing the parameters in terms of irreducible constituents of the discrete spectrum of general linear groups. For quasi-split classical groups this approach was taken by Moeglin in [Mœg08].

3.1. **Arthur parameters.** Let μ be a Galois self-dual cuspidal automorphic representation of $GL_m(\mathbb{A}_E)$. One of the crucial results in Mok's proof of endoscopic classification of representations in the discrete spectrum for quasi-split unitary groups is the uniqueness (up to equivalence) of the twisted endoscopic datum associated to μ. This is the content of [Mok, Thm. 2.4.2]. In fact, this unique endoscopic datum is simple, thus, determining a unique sign $\kappa \in \{\pm 1\}$ attached to μ. The parity of the endoscopic datum associated to μ is then defined as $\kappa(-1)^{m-1}$ (cf. [Mok, Sect. 2.4]). Using parity we make the following definition as in [Mok, Thm. 2.5.4] (see also [GGP12]).

Definition 3.1. Let μ be a Galois self-dual cuspidal automorphic representation of $GL_m(\mathbb{A}_E)$. We say that μ is Galois orthogonal (resp. Galois symplectic), if the parity of the unique twisted endoscopic datum associated to μ is $+1$ (resp. -1).

It turns out, as also proved by Mok, that this definition can be rephrased in terms of poles at $s = 1$ of the Asai L-function $L(s, \mu, r_A)$ attached to μ.

Theorem 3.2 ([Mok, Thm. 2.5.4 (a)])**.** *Let μ be a Galois self-dual cuspidal automorphic representation of $GL_m(\mathbb{A}_E)$. Then, μ is Galois orthogonal (resp. Galois symplectic) if and only if the Asai L-function $L(s, \mu, r_A)$ (resp. the twisted Asai L-function $L(s, \mu \otimes \widehat{\delta}, r_A)$) has a pole at $s = 1$, where $\widehat{\delta}$ is any extension to $\mathbb{A}_E^\times/E^\times$ of the quadratic character $\delta_{E/F}$ of $\mathbb{A}_F^\times/F^\times$ attached to E/F by class field theory.*

We are now ready to define global Arthur parameters for the quasi-split unitary group U_N in N variables. We in fact define the square-integrable Arthur parameters, which, according to [Mok, Thm. 2.5.2], parameterize global Arthur packets contributing to the discrete automorphic spectrum of $U_N(\mathbb{A}_F)$. These parameters depend on the choice of certain character of $\mathbb{A}_E^\times$, trivial on $E^\times$, that defines an L-embedding of the L-group of U_N into the L-group of G_N (cf. [Mok, Sect. 2.1]). Roughly speaking, this character determines whether we view parameters as the stable or twisted base change of a representation in the discrete spectrum. Of course, the decomposition of the discrete spectrum is independent of that choice, and we take it in this paper to be the trivial character of $\mathbb{A}_E^\times$, and suppress it from notation (see [Mok, Thm. 2.5.2]). The reason why Mok considers all possible characters is that they are all required for the induction argument in the proof of endoscopic classification.

Definition 3.3 (Arthur parameters)**.** As before, let U_N be the quasi-split unitary group in N variables given by a quadratic extension E/F of number fields. The set $\Psi_2(U_N)$ of square-integrable global Arthur parameters for U_N is defined as the set of all *unordered* formal sums of formal tensor products of the form

$$\psi = \big(\mu_1 \boxtimes \nu(n_1)\big) \boxplus \cdots \boxplus \big(\mu_\ell \boxtimes \nu(n_\ell)\big),$$

such that

(i) μ_i is a Galois self-dual cuspidal automorphic representation of $GL_{m_i}(\mathbb{A}_E)$, that is, $\mu_i \cong \widetilde{\mu}_i^\theta$,
(ii) n_i is a positive integer, and $\nu(n_i)$ is the unique n_i-dimensional irreducible algebraic representation of $SL_2(\mathbb{C})$,
(iii) $m_1n_1 + \cdots + m_\ell n_\ell = N$,
(iv) for $i \neq j$, we have $\mu_i \ncong \mu_j$ or $n_i \neq n_j$, that is, the formal sum ψ is multiplicity free,
(v) representation μ_i is Galois orthogonal (resp. Galois symplectic) if and only if integers n_i and N are of the same parity (resp. different parity).

According to Theorem 3.2, condition (v) is equivalent to the condition

(v′) representation μ_i is such that the Asai L-function $L(s, \mu_i, r_A)$ (resp. the twisted Asai L-function $L(s, \mu_i \otimes \widehat{\delta}, r_A)$) has a pole at $s = 1$ if and only if integers n_i and N are of the same parity (resp. different parity).

3.2. **Arthur packets.** We proceed, following [Mok], to define the local and global Arthur packet associated to a global Arthur parameter $\psi \in \Psi_2(U_N)$. Every global Arthur parameter $\psi \in \Psi_2(U_N)$ gives rise, as in [Mok, Sect. 2.3], to a local Arthur parameter ψ_v for every place v of F. The local Arthur packet Π_{ψ_v} is a finite multi-set of unitary irreducible representations of $U_N(F_v)$ associated to ψ_v in [Mok, Thm. 2.5.1] and the discussion following it. There is a canonical mapping from Π_{ψ_v} to the character group of a certain finite group $\mathcal{S}_{\psi_v}$ attached to ψ_v (for a definition see [Mok, Sect. 2.2]). For $\pi_v \in \Pi_{\psi_v}$, we denote the corresponding character by η_{π_v}. If $U_N(F_v)$ and π_v are unramified, then η_{π_v} is the trivial character. We are skipping here the details, because our main interest is only in unramified places.

The global Arthur packet Π_ψ associated to $\psi \in \Psi_2(U_N)$ is defined as

$$\Pi_\psi = \left\{ \otimes_v' \pi_v \,:\, \pi_v \in \Pi_{\psi_v} \text{ and } \eta_{\pi_v} \text{ is trivial for almost all } v \right\}.$$

The global packets Π_ψ for all $\psi \in \Psi_2(U_N)$ contain all representations that can possibly appear in the decomposition of the discrete spectrum on $U_N(\mathbb{A}_F)$. There is a subtle further condition identifying elements of Π_ψ that indeed appear in the discrete spectrum (for a precise formulation see [Mok, Thm. 2.5.2]). We do not recall this condition, because for our purposes it is sufficient to work with the full packets Π_ψ.

We now compare a representation in the discrete spectrum on $U_N(\mathbb{A}_F)$ and its Arthur parameter at unramified places. Through the application to residual representations supported in the Siegel maximal parabolic subgroup, this turns out to be crucial for the proof of holomorphy of the Asai L-function inside the critical strip. Given

$$\psi = \left(\mu_1 \boxtimes \nu(n_1)\right) \boxplus \cdots \boxplus \left(\mu_\ell \boxtimes \nu(n_\ell)\right) \in \Psi_2(U_N),$$

with notation as in Definition 3.3, let S be a finite set of places of F, containing all archimedean places and all non-archimedean places ramified in E, and such that for all places w of E lying above some $v \notin S$ all $\mu_{i,w}$ are unramified. Then, for $v \notin S$, we attach to ψ a Frobenius-Hecke conjugacy class

$$c_v(\psi) = \begin{cases} \bigoplus_{i=1}^\ell \left(c(\mu_{i,w}) \otimes c_w(\nu(n_i))\right), & \text{if } v \text{ is inert and } v \mid w, \\ \left(\bigoplus_{i=1}^\ell \left(c(\mu_{i,w_1}) \otimes c_{w_1}(\nu(n_i))\right), \bigoplus_{i=1}^\ell \left(c(\mu_{i,w_2}) \otimes c_{w_2}(\nu(n_i))\right)\right), & \text{if } v \text{ splits into } w_1, w_2, \end{cases}$$

viewed as a semi-simple conjugacy class in the L-group of G_N over F_v, where $c(\mu_{i,w}) \in GL_{m_i}(\mathbb{C})$ is the Satake parameter, and

$$c_w(\nu(n_i)) = \mathrm{diag}\Big(q_w^{\frac{n_i-1}{2}}, q_w^{\frac{n_i-3}{2}}, \dots, q_w^{-\frac{n_i-1}{2}}\Big)$$

with q_w the cardinality of the residue field of E_w. Observe that $q_w = q_v^2$ if v is inert in E, and $q_{w_1} = q_{w_2} = q_v$ if v splits in E. The conjugacy classes $c_v(\psi)$ for $v \notin S$ may be viewed as the Satake parameters of the unramified constituents at places w of E lying above v of the induced representation

$$\mathrm{Ind}_{R(\mathbb{A}_E)}^{GL_N(\mathbb{A}_E)} \left(\begin{array}{l} \mu_1|\det|^{\frac{n_1-1}{2}} \otimes \mu_1|\det|^{\frac{n_1-3}{2}} \otimes \cdots \otimes \mu_1|\det|^{-\frac{n_1-1}{2}} \otimes \\ \mu_2|\det|^{\frac{n_2-1}{2}} \otimes \mu_2|\det|^{\frac{n_2-3}{2}} \otimes \cdots \otimes \mu_2|\det|^{-\frac{n_2-1}{2}} \otimes \cdots \\ \otimes \mu_\ell|\det|^{\frac{n_\ell-1}{2}} \otimes \mu_\ell|\det|^{\frac{n_\ell-3}{2}} \otimes \cdots \otimes \mu_\ell|\det|^{-\frac{n_\ell-1}{2}} \end{array} \right)$$

where R is the standard parabolic subgroup of GL_N with the Levi factor $GL_{m_1} \times \cdots \times GL_{m_1} \times GL_{m_2} \times \cdots \times GL_{m_2} \times \cdots \times GL_{m_\ell} \times \cdots \times GL_{m_\ell}$ with n_i copies of GL_{m_i} in the product, and μ_i are unramified at v.

On the other hand, let $\pi \cong \otimes_v' \pi_v$ be an irreducible automorphic representation appearing in the discrete spectrum on $U_N(\mathbb{A}_F)$. Let S' be a finite set of places of F, containing all archimedean places, and such that for $v \notin S'$, we have that $U_N(F_v)$ and π_v are unramified. Then, for $v \notin S'$, the Satake isomorphism gives a Frobenius-Hecke conjugacy class $c(\pi_v)$ in the local L-group of U_N over F_v. However, we may view $c(\pi_v)$ as a conjugacy class in the local L-group of G_N through the stable base change map of L-groups. This is consistent with our choice of the trivial character in the definition of Arthur parameters.

According to the preliminary comparison of spectral sides of the trace formulas for U_N and the twisted trace formula for GL_N, carried out in [Mok, Sect. 4.3] (see also [Art13, Sect. 3.4]), for every irreducible automorphic representation π of $U_N(\mathbb{A}_F)$ appearing in the discrete spectrum, there is a unique corresponding parameter $\psi \in \Psi_2(U_N)$ such that the Frobenius-Hecke conjugacy classes $c_v(\psi)$ attached to ψ coincide at almost all places with the classes $c(\pi_v)$ attached to π. This observation is the key to the following proposition.

Remark 3.4. Strictly speaking the preliminary comparison of trace formulas gives unique ψ in a larger set of parameters $\Psi(U_N)$ (see [Mok] for a definition), but the full proof of endoscopic classification shows that such ψ belongs to $\Psi_2(U_N)$.

Proposition 3.5. *Let P be the Siegel maximal proper parabolic F-subgroup of U_{2n}. Let σ be a cuspidal automorphic representation of its Levi factor $M_P(\mathbb{A}_F) \cong GL_n(\mathbb{A}_{\mathbb{E}})$. If the induced representation*

$$\mathrm{Ind}_{P(\mathbb{A}_F)}^{U_{2n}(\mathbb{A}_F)} \left(\sigma \otimes |\det|_{\mathbb{A}_E}^s\right)$$

has a constituent in the discrete spectrum of $U_{2n}(\mathbb{A}_F)$ for some $s > 0$, then its Arthur parameter is

$$\psi = \sigma \boxtimes \nu(2),$$

and in particular $s = 1/2$ and σ is Galois self-dual.

Proof. Since an automorphic representation is unramified at almost all places, the local component of an irreducible constituent π of the induced representation

$$\mathrm{Ind}_{P(\mathbb{A}_F)}^{U_{2n}(\mathbb{A}_F)} \left(\sigma \otimes |\det|_{\mathbb{A}_E}^s\right)$$

belonging to the discrete spectrum is the unramified representation with the Satake parameter, viewed as a conjugacy class in the L-group of G_{2n} as above,

$$c(\pi_v) = \begin{cases} c(\sigma_w) \otimes \mathrm{diag}\Big(q_w^s, q_w^{-s}\Big), & \text{if } v \text{ is inert and } v \mid w, \\ \Big(c(\sigma_{w_1}) \otimes \mathrm{diag}\Big(q_{w_1}^s, q_{w_1}^{-s}\Big), c(\sigma_{w_2}) \otimes \mathrm{diag}\Big(q_{w_2}^s, q_{w_2}^{-s}\Big)\Big), & \text{if } v \text{ splits into } w_1, w_2, \end{cases}$$

for almost all places v of F. Recall that $q_w = q_v^2$ if v is inert, and $q_{w_1} = q_{w_2} = q_v$ if v splits. We may also view $c(\pi_v)$ as the Satake parameter of the unramified constituent of the local components at places w of E lying over v of the induced representation

$$\mathrm{Ind}_{Q(\mathbb{A}_E)}^{GL_{2n}(\mathbb{A}_E)} \Big(\sigma|\det|_{\mathbb{A}_E}^s \otimes \sigma|\det|_{\mathbb{A}_E}^{-s}\Big),$$

where Q is the standard parabolic subgroup of GL_{2n} with the Levi factor $GL_n \times GL_n$.

By the observation made just before the statement of the proposition, these Frobenius-Hecke conjugacy classes $c(\pi_v)$, viewed as conjugacy classes in the L-group of G_{2n}, should match at almost all places the conjugacy classes $c_v(\psi)$ attached to the Arthur parameter $\psi \in \Psi_2(U_N)$ parameterizing π. As mentioned above, these $c_v(\psi)$ may be viewed as Satake parameters of the unramified constituent at v of certain induced representation of $GL_{2n}(\mathbb{A}_E)$. However, by the strong multiplicity one for general linear groups [JS81, Thm. 4.4], matching of Satake parameters at almost all places for induced representations of $GL_N(\mathbb{A}_E)$ implies that the inducing data for these representations are associate. Since Q is self-associate, this means that the parabolic subgroup R determined by ψ as above must be Q, and thus that ψ is of the form

$$\psi = \sigma \boxtimes \nu(k),$$

where $k = 2s + 1$. Since $k = 2$ by condition (iii) in Definition 3.3, it follows that $s = 1/2$. As σ appears in ψ it is necessarily Galois self-dual. $\square$

4. Holomorphy and non-vanishing of Asai L-functions

In this section we prove the analytic properties of the Asai L-functions as a consequence of Mok's endoscopic classification of automorphic representations of a quasi-split unitary group [Mok].

4.1. **Analytic properties of Eisenstein series.** The first task is to determine the poles of Eisenstein series $E(f, s)$ for $Re(s) > 0$. We now consider only the case of even quasi-split unitary group U_{2n}.

Recall that for a cuspidal automorphic representation σ of $GL_n(\mathbb{A}_E)$, we let σ^θ denote σ conjugated by the non-trivial Galois automorphism $\theta \in Gal(E/F)$. We say that σ is Galois self-dual if it is isomorphic to $\widetilde{\sigma}^\theta$, where $\widetilde{\sigma}$ is the contragredient of σ.

Theorem 4.1. *Let σ be a cuspidal automorphic representation of the Levi factor $M_P(\mathbb{A}_F) \cong GL_n(\mathbb{A}_E)$ in U_{2n}. Then, the Eisenstein series $E(f, s)$ on $U_{2n}(\mathbb{A}_F)$, constructed as in Sect. 2.2 from functions f in the representation space W_σ on which induced representations $I(s, \sigma)$ are realized for all s, is*

1. *holomorphic for $Re(s) \geq 0$, if σ is not Galois self-dual,*
2. *holomorphic for $Re(s) \geq 0$, except for a possible simple pole at $s = 1/2$, if σ is Galois self-dual.*

Proof. The Eisenstein series is holomorphic on the imaginary axis $Re(s) = 0$ (see [MW95, Sect. IV.1.11]). Hence, we may assume $Re(s) > 0$. Suppose that the Eisenstein series $E(f, s)$ on $U_{2n}(\mathbb{A}_F)$ has a pole at $s = s_0 > 0$ for some $f \in W_\sigma$ in the notation of Section 2. Since $s_0 > 0$, the residues at $s = s_0$ of $E(f, s)$ when $f \in W_\sigma$ span a residual automorphic representation of $U_{2n}(\mathbb{A}_F)$. But this residual representation is a constituent of the induced representation

$$\mathrm{Ind}_{P(\mathbb{A}_F)}^{U_{2n}(\mathbb{A}_F)} \left(\sigma \otimes |\det|_E^{s_0}\right).$$

By Proposition 3.5, its Arthur parameter is

$$\psi = \sigma \boxtimes \nu(2),$$

where σ is Galois self-dual and $s_0 = 1/2$. Therefore, the Eisenstein series $E(f, s)$ is holomorphic for $Re(s) > 0$, except for a possible pole at $s = 1/2$ if σ is Galois self-dual, as claimed. The possible pole is simple, by the general theory of Eisenstein series [MW95, Sect. IV.1.11] $\square$

Remark 4.2. A significant part of Theorem 4.1 can be proved in a different way, without using Mok's work on the Arthur classification for unitary groups [Mok], which is based on the trace formula, and still depends on the stabilization of the twisted trace formula for GL_n.

For instance, if σ is not Galois self-dual, the following general argument provides holomorphy of the Eisenstein series for $Re(s) > 0$. By [HC68], see also [MW95, Sect. IV.3.12], a necessary condition for the Eisenstein series $E(f, s)$ to have a pole for $Re(s) > 0$ and some $f \in W_\sigma$ is that $\sigma^{w_0} \cong \sigma$. But in our case, $\sigma^{w_0} = \widetilde{\sigma}^\theta$, so that $E(f, s)$ is holomorphic for $Re(s) > 0$ and all $f \in W_\sigma$ if σ is not Galois self-dual.

If σ is Galois self-dual, there is a unitarity argument, which gives the analytic behavior of the Eisenstein series for $Re(s) \geq 1/2$. However, the critical strip $0 < Re(s) < 1/2$ remains out of reach. For completeness, we include this argument in Sect. 4.3 below.

4.2. **Analytic properties of Asai L-functions.** The following theorem describes completely the analytic properties of the Asai L-functions attached to a cuspidal automorphic representation σ of $GL_n(\mathbb{A}_E)$. It is the main result of the paper.

Theorem 4.3. *Let σ be a cuspidal automorphic representation of $GL_n(\mathbb{A}_E)$. Let $L(s, \sigma, r_A)$ (respectively, $L(s, \sigma \otimes \widehat{\delta}, r_A)$) be the Asai (respectively, twisted Asai) L-function attached to σ, where $\widehat{\delta}$ is any extension to $\mathbb{A}_E^\times/E^\times$ of the quadratic character of $\mathbb{A}_F^\times/F^\times$ attached to the extension E/F by class field theory.*

1. *If σ is* not *Galois self-dual, i.e., $\sigma \not\cong \widetilde{\sigma}^\theta$, then $L(s, \sigma, r_A)$ is entire. It is non-zero for $Re(s) \geq 1$ and $Re(s) \leq 0$.*
2. *If σ is Galois self-dual, i.e., $\sigma \cong \widetilde{\sigma}^\theta$, then*
   - (a) *$L(s, \sigma, r_A)$ is entire, except for possible simple poles at $s = 0$ and $s = 1$, and non-zero for $Re(s) \geq 1$ and $Re(s) \leq 0$;*
   - (b) *exactly one of the L-functions $L(s, \sigma, r_A)$ and $L(s, \sigma \otimes \widehat{\delta}, r_A)$ has simple poles at $s = 0$ and $s = 1$, while the other is holomorphic at those points.*

Proof. The idea of the proof goes back to [Sha81], [Sha88]. The proof of holomorphy is based on Theorem 2.1, which relates the poles of Eisenstein series to the Asai L-functions, and Theorem 4.1 providing the analytic behavior of the Eisenstein series. The non-vanishing, on the other hand, follows from considering the non-constant term of the Eisenstein series as in [Sha81], see also [Sha10, Sect. 7], and using Theorem 4.1 again. It is sufficient to prove the claims for $Re(s) \geq 1/2$, due to the functional equation for Asai L-functions.

We begin with the proof of holomorphy. Consider first the case of σ not Galois self-dual. According to Theorem 4.1, the Eisenstein series attached to σ is holomorphic for $Re(s) > 0$. Assume that $L(s, \sigma, r_A)$ has a pole for $s = s_0 > 0$. Since the poles of $E(f, s)$ for $Re(s) > 0$ coincide, according to Theorem 2.1, with the poles of the ratio

$$\frac{L(2s, \sigma, r_A)}{L(1 + 2s, \sigma, r_A)}, \tag{$*$}$$

the pole of the numerator at $2s = s_0 > 0$ should be canceled by a pole in the denominator. Thus, $L(z, \sigma, r_A)$ should have a pole at $z = s_0 + 1$. Repeating this argument, we obtain a sequence of poles of the Asai L-function of the form $s_0 + M$, where M is any non-negative integer. This is a contradiction, because $L(s, \sigma, r_A)$ is holomorphic in the right half-plane of absolute convergence of the defining product. Thus, we proved that $L(s, \sigma, r_A)$ is entire.

Consider now the case of σ Galois self-dual. By Theorem 4.1, the Eisenstein series $E(f, s)$ attached to σ is holomorphic for $Re(s) > 0$, except for a possible simple pole at $s = 1/2$. The same argument as in the previous case implies that $L(z, \sigma, r_A)$ is holomorphic for $Re(z) > 0$, except for $z = 1$ if the Eisenstein series has a pole at $s = 1/2$.

To prove that a possible pole of $L(z, \sigma, r_A)$ at $z = 1$ is at most simple, we again apply a similar argument. Suppose $E(f, s)$ has a pole at $s = 1/2$. It is simple by Theorem 4.1. If $L(z, \sigma, r_A)$ had a higher order pole at $z = 2s = 1$, then Theorem 2.1 would imply that there is a pole in the denominator of the ratio of Asai L-functions in Equation $(*)$. But this would mean that the Asai L-function has a pole at $z + 1 = 2$. The Eisenstein series is holomorphic at $s = 1$, so that the same argument as before gives a sequence of poles at all positive integers, which is a contradiction.

For non-vanishing, consider the non-constant term $E(f, s)_\psi$ of the Eisenstein series $E(f, s)$ with respect to a fixed non-trivial additive character ψ of $F \backslash \mathbb{A}_F$. According to [Sha10, Thm. 7.1.2], we have

$$E(f, s)_\psi(e) = \frac{1}{L^S(1 + 2s, \sigma, r_A)} \cdot \prod_{v \in S} W_v(e_v),$$

where e and e_v are the identity matrices, W_v is the ψ_v-Whittaker function attached to f via a Jacquet integral, S is a finite set of places, containing all archimedean places, outside which $U_{2n}(F_v)$, σ_v and ψ_v are all unramified, and $L^S(z, \sigma, r_A)$ is the partial Asai L-function attached to σ. As in [Sha10, Sect. 7.2], there is a choice of $f \in W_\sigma$ such that $W_v(e_v) \neq 0$ for all $v \in S$. Thus, every zero of $L^S(1 + 2s, \sigma, r_A)$ for $Re(s) \geq 0$, i.e. $Re(1 + 2s) \geq 1$ would give a pole of the non-constant term $E(f, s)_\psi$. However, by Theorem 4.1, the Eisenstein series $E(f, s)$, and thus $E(f, s)_\psi$ as well, is holomorphic for $Re(s) \geq 0$, except for a possible pole at $s = 1/2$, which may occur only if σ is Galois self-dual. Hence, $L^S(z, \sigma, r_A)$ has no zeroes for $Re(z) \geq 1$, except possibly for $z = 1$. Since the local L-functions are non-vanishing, the same holds for the complete Asai L-function $L(z, \sigma, r_A)$.

For σ Galois self-dual, the non-vanishing of $L(z, \sigma, r_A)$ at the remaining point $z = 1$ follows from the identity

$$L(s, \sigma \times \sigma^\theta) = L(s, \sigma, r_A) L(s, \sigma, r_A \otimes \delta_{E/F}), \tag{$**$}$$

where $L(s, \sigma \times \sigma^\theta)$ is the Rankin-Selberg L-function, and recall that the twisted Asai L-function equals

$$L(s, \sigma, r_A \otimes \delta_{E/F}) = L(s, \sigma \otimes \widehat{\delta}, r_A).$$

See [Gol94] for these identities. The poles of the Rankin-Selberg L-function $L(s, \sigma \times \sigma^\theta)$ are known from [JS81]. For σ Galois self-dual it has a simple pole at $s = 1$. Since $\sigma \otimes \widehat{\delta}$ is Galois self-dual as well, we already proved that both Asai L-functions on the right-hand side of $(**)$ have at most simple pole at $s = 1$. Hence, they are both non-zero at $s = 1$, and exactly one of them has a simple pole at $s = 1$, as claimed. □

Remark 4.4. Once the holomorphy of the Asai and twisted Asai L-function is known at some s_0 with $Re(s_0) > 0$, the argument using the Rankin-Selberg L-function at the end of this proof can be applied directly to obtain non-vanishing. However, the result of Jacquet-Shalika [JS81] providing analytic properties of the Rankin-Selberg L-functions is very deep, and we preferred to give an argument using non-constant term of the Eisenstein series whenever possible.

4.3. **Holomorphy of Eisenstein series using a unitarity argument.** We now give a different proof that the Eisenstein $E(f, s)$, attached to a Galois self-dual cuspidal automorphic representation σ of $GL_n(\mathbb{A}_F)$ as above, is holomorphic for $Re \geq 1/2$, except for a possible simple pole at $s = 1/2$.

It is sufficient to prove that $E(f, s)$ is holomorphic for $Re(s) > 1/2$. Indeed, since we always normalize σ to be trivial on $A_P(F_\infty)^\circ$, the poles of the Eisenstein series are real. Hence, the only possible pole for $Re(s) = 1/2$ is at $s = 1/2$. It is at most simple pole, because all poles of Eisenstein series inside the closure of the positive Weyl chamber are without multiplicity [MW95, Sect. IV.1.11].

Suppose that there is a simple pole of $E(f, s)$ at $s = s_0 > 1/2$. We follow an idea of Kim [Kim00] based on the fact that residual representations are unitary. The space of residues of $E(f, s)$ at $s = s_0$ is a residual representation of $U_{2n}(\mathbb{A}_F)$, which is a constituent of the induced representation

$$I(s_0, \sigma) = \operatorname{Ind}_{P(\mathbb{A}_F)}^{U_{2n}(\mathbb{A}_F)} \left(\sigma |\det|_E^{s_0}\right).$$

In particular, this residual representation is unitary, so that the induced representation should have a unitary constituent. But then the local induced representation at every place v should have a unitary subquotient. Let v be a split non-archimedean place of F such that σ_v is unramified. The local induced representation at v is isomorphic to

$$I(s_0, \sigma_v) \cong \operatorname{Ind}_{P(F_v)}^{GL_{2n}(F_v)} \left(\sigma_{w_1} |\det|_{F_v}^{s_0} \otimes \widetilde{\sigma}_{w_2} |\det|_{F_v}^{-s_0}\right),$$

where w_1 and w_2 are the two places of E lying above v. Since σ_{w_1} and σ_{w_2} are unramified unitary generic representations of $GL_n(F_v)$, according to [Tad86], they are fully induced representations of the form

$$\begin{aligned}
\sigma_{w_1} &\cong \operatorname{Ind}_{B_n(F_v)}^{GL_n(F_v)} \left(\mu_1 | \, |^{\alpha_1} \otimes \cdots \otimes \mu_k | \, |^{\alpha_k} \otimes \chi_1 \otimes \cdots \otimes \chi_l \otimes \mu_k | \, |^{-\alpha_k} \otimes \cdots \otimes \mu_1 | \, |^{-\alpha_1}\right), \\
\widetilde{\sigma}_{w_2} &\cong \operatorname{Ind}_{B_n(F_v)}^{GL_n(F_v)} \left(\mu_1' | \, |^{\beta_1} \otimes \cdots \otimes \mu_{k'}' | \, |^{\beta_{k'}} \otimes \chi_1' \otimes \cdots \otimes \chi_{l'}' \otimes \mu_{k'}' | \, |^{-\beta_{k'}} \otimes \cdots \otimes \mu_1' | \, |^{-\beta_1}\right),
\end{aligned}$$

where B_n is a Borel subgroup of GL_n, the exponents satisfy $0 < \alpha_k < \cdots < \alpha_1 < 1/2$ and $0 < \beta_{k'} < \cdots < \beta_1 < 1/2$, and μ_i, μ'_i, χ_j, χ'_j are unramified unitary characters of $F_v^\times$. Hence,

$$I(s_0,\sigma_v) \cong \mathrm{Ind}_{B_{2n}(F_v)}^{GL_{2n}(F_v)} \left(\begin{array}{c} \mu_1|\,|^{s_0+\alpha_1} \otimes \cdots \otimes \mu_k|\,|^{s_0+\alpha_k} \otimes \chi_1|\,|^{s_0} \otimes \cdots \otimes \chi_l|\,|^{s_0} \otimes \\ \mu_k|\,|^{s_0-\alpha_k} \otimes \cdots \otimes \mu_1|\,|^{s_0-\alpha_1} \otimes \\ \mu'_1|\,|^{-s_0+\beta_1} \otimes \cdots \otimes \mu'_{k'}|\,|^{-s_0+\beta_{k'}} \otimes \chi'_1|\,|^{-s_0} \otimes \cdots \otimes \chi'_{l'}|\,|^{-s_0} \otimes \\ \mu'_{k'}|\,|^{-s_0-\beta_{k'}} \otimes \cdots \otimes \mu'_1|\,|^{-s_0-\beta_1} \end{array} \right).$$

According to the description of the unitary dual of $GL_{2n}(F_v)$ [Tad86], this representation would have a unitary subquotient, only if all the exponents whose absolute value is not smaller than $1/2$, induced with another character to a representation of $GL_2(F_v)$, give a reducible representation with a unitary quotient of Speh type. However, this is possible only if for every such exponent that is not less than $1/2$ in absolute value, there is another exponent such that their difference is exactly 1.

Having this in mind, consider the largest exponent in the above induced representation. We write this exponent as $s_0 + \alpha_1$, and allow the possibility $\alpha_1 = 0$, which happens in the case $k = 0$ as there are no α_i's. There should be another exponent of the form $-s_0 \pm \beta$, where $\beta = \beta_j$ for some j or $\beta = 0$, such that

$$(s_0 + \alpha_1) - (-s_0 \pm \beta) = 1.$$

But this implies

$$2s_0 + \alpha_1 \mp \beta = 1,$$

which is possible for $s_0 > 1/2$ only if the sign of β is minus and $\alpha_1 < \beta$. As beta is certainly not greater than the largest of β_j's, it follows that necessarily $\alpha_1 < \beta_1$. However, considering the smallest exponent in the induced representation, that is, $-s_0 - \beta_1$, where again β_1 is set to zero if $l = 0$, we obtain the opposite inequality, $\beta_1 < \alpha_1$. This is a contradiction, proving that $I(s_0,\sigma_v)$ has not a unitary subquotient for $s_0 > 1/2$, and therefore, the Eisenstein series $E(f,s)$ has no pole for $Re(s) > 1/2$, as claimed.

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