CORRESPONDENCE BETWEEN THE RESIDUAL SPECTRA OF RANK TWO SPLIT CLASSICAL GROUPS AND THEIR INNER FORMS

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Introduction

In this paper the correspondence between the residual spectra of the classical rank two split groups SO_4 and Sp_4 and their inner forms G_1' and H_1' is obtained. As algebraic groups over an algebraic number field, hermitian quaternionic groups G_1' and H_1' are defined in Section 1. They are non–quasi split. The problem of comparing the residual spectra of split groups and their inner forms is still open even for the general linear group as mentioned in [3] and Section 25 of [2]. In this paper, the correspondence is obtained for the low rank case by the explicit decomposition of the residual spectra of all the groups involved.

For quasi–split groups the residual spectrum has been considered by several authors. Among others Mœglin and Walspurger in [21], Mœglin in [18], [19] and [20], Kim in [11], [12] and [15], Žampera in [29], Kon–No in [16]. All those papers use Langland–Shahidi method described in [25] and [26] for normalization of standard intertwining operators.

For the split group Sp_4 the residual spectrum is decomposed into irreducible constituents by Kim in [11] and in this paper we use his result recalled in Theorem 3.4. Decomposition of the residual spectrum for the split group SO_4 is obtained by the same method in Theorem 3.3 of this paper.

However, groups G_1' and H_1' are out of scope of Langlands–Shahidi method since they are non–quasi split. For non–quasi split groups the residual spectrum was considered in author's papers [6] and [7]. In those papers we developed a new technique for normalization of standard intertwining operators which is applied in this paper. It is based on Jacquet–Langlands correspondence explained in [5] and comparison of Plancherel measures as in [24].

Although, in principle, the residual spectra of G_1' and H_1' could be decomposed using Arthur's trace formula explained in [2], we use more direct approach of Langlands spectral theory explained in [17] and [22]. Decompositions of the residual spectra for groups G_1' and H_1' are obtained in Theorems 3.1 and 3.2. Already for these low rank cases, the results show certain ambiguities of hermitian quaternionic groups such as condition on non–triviality of local components at all non–split places in Theorem 3.2.

Finally, as a consequence of decomposition Theorems, we obtain correspondences between the residual spectra of G_1' and SO_4 in Corollary 3.5 and between the residual spectra of H_1' and Sp_4 in Corollary 3.6.

The paper is divided into three Sections. In the first Section we introduce notation and recall basic structural facts for the groups involved. In the second Section we collect global normalizing factors required in calculation and obtained in [6] and [7]. Finally, in the third Section we decompose the residual spectra and give the correspondences.

This paper is prepared for the proceedings of the conference Functional Analysis IX held in June, 2005 in Dubrovnik, Croatia. I would like to thank the organizers for the opportunity to give a lecture there. I would like to thank G. Muić for many useful discussions and constant help during preparation of this paper. I would like to thank M. Tadić for supporting my research and his interest in my work. The conversations with H. Kim and E. Lapid were useful in clarifying several issues in automorphic forms and with I. Badulescu in representation theory of GL_n over division algebras. Also I would like to thank my friend M. Hanzer for many useful conversations on the local representation theory of hermitian quaternionic groups. And finally, I would like to thank my wife Tiki for being the best wife in the world.

1. Group Structure and Notation

In this Section we define groups considered in this paper, review their structure and introduce notation. Let k be an algebraic number field, k_v its completion at place v and $\mathbb{A}$ its ring of adeles. Let D be a quaternion algebra central over k and τ the usual involution fixing the center of D. Then D splits at all but finitely many places v of k, i.e. at those places the completion $D \otimes_k k_v$ is isomorphic to the additive group $M(2, k_v)$ of 2×2 matrices with coefficients in k_v. At finitely many places v of k where D is non–split the completion $D \otimes_k k_v$ is isomorphic to the quaternion algebra D_v central over k_v. The finite set of places of k where D is non–split is denoted by S. The cardinality of S, denoted by $|S|$, is even for every D.

The algebraic group over k of invertible elements of D is denoted GL'_1. At a split place $v \notin S$ it is isomorphic to $GL'_1(k_v) \cong GL_2(k_v)$, where GL_2 is a split group over k of invertible 2×2 matrices. At a non–split place $v \in S$ it is isomorphic to $GL'_1(k_v) \cong D_v^\times$.

Let $\det'$ denote the reduced norm of the simple algebra $D \otimes_k \mathbb{A}$ and $\det'_v$ the corresponding reduced norm at place v. If $v \notin S$ is split, then $\det'_v = \det_v$ is just the determinant for 2×2 matrices, while if $v \in S$ is non–split, then $\det'_v$ is the reduced norm of the quaternion algebra D_v. The absolute value of the reduced norm $\det'$ and $\det'_v$ is denoted by ν.

Let V be a hyperbolic plane over D, i.e. a two–dimensional right vector space over D equipped with a hermitian form $(\cdot, \cdot)$ defined in the basis $\{e_1, e_2\}$ of V by

$$(e_1x_1 + e_2x_2, e_1x'_1 + e_2x'_2) = \tau(x_1)x'_2 + \epsilon\tau(x_2)x'_1$$

for all $x_1, x_2, x'_1, x'_2 \in D$, where $\epsilon \in \{\pm 1\}$. The group of isometries of the form $(\cdot, \cdot)$ regarded as a reductive algebraic group defined over k will be denoted by G'_1 if $\epsilon = -1$ and by H'_1 if $\epsilon = 1$. Then, G'_1 is an inner form of the split group SO_4, while H'_1 is an inner form of the split group Sp_4. Hence $G'_1(k_v) \cong SO_4(k_v)$ and $H'_1(k_v) \cong Sp_4(k_v)$ for every place $v \notin S$.

Both, G'_1 and H'_1, have only one standard proper parabolic subgroup defined over k. Abusing the notation, we let P'_0 denote that subgroup for both groups. The Levi factor M'_0 of P'_0 is isomorphic to GL'_1. For the split groups SO_4 and Sp_4, let P_0 denote the standard parabolic subgroup with Levi factor $M_0 \cong GL_2$ such that P'_0 is an inner form of P_0. The maximal split torus isomorphic to $GL_1 \times GL_1$ for those split groups is denoted by T.

The Weyl groups for G'_1 and H'_1 with respect to the maximal split torus are $W' = \{1, w\}$, where w is the unique nontrivial element. For the corresponding split case Levi factor $M_0 \cong GL_2$ in SO_4 or Sp_4 let $W(M_0)$ denote the subgroup of the Weyl group W consisting of elements fixing the Levi factor M_0. Then $W' \cong W(M_0)$ and again by w we denote the nontrivial element of $W(M_0)$.

Let Δ be the set of simple roots of split group SO_4 or Sp_4. If e_i is the character of T defined by $e_i(t_1,t_2)=t_i$ for $(t_1,t_2)\in T\cong GL_1\times GL_1$, then

$$\Delta=\begin{cases}\{e_1-e_2,e_1+e_2\}, & \text{for the group } SO_4,\\ \{e_1-e_2,2e_2\}, & \text{for the group } Sp_4.\end{cases}$$

Let w_1 in both cases be the simple reflection with respect to the root e_1-e_2 and w_2 the simple reflection with respect to e_1+e_2 for SO_4 and $2e_2$ for Sp_4. Then the Weyl group equals

$$W=\begin{cases}\{1,w_1,w_2,w_1w_2\}, & \text{for the group } SO_4,\\ \{1,w_1,w_2,w_1w_2,w_2w_1,w_1w_2w_1,w_2w_1w_2,w_1w_2w_1w_2\}, & \text{for the group } Sp_4.\end{cases}$$

Let $\mathfrak{a}^*_{M_0,\mathbb{C}}\cong X(M_0')\otimes_{\mathbb{Z}}\mathbb{C}$ denote the complexification of the $\mathbb{Z}$–module $X(M_0')$ of k–rational characters of M_0'. It is one–dimensional and we identify it with $\mathbb{C}$ taking for the basis the reduced norm on GL_1'. In the split case of M_0 in SO_4 and Sp_4 the complexification of the $\mathbb{Z}$–module $X(M_0)$ of k–rational characters of M_0 is again $\mathfrak{a}^*_{M_0,\mathbb{C}}$. For the torus T in SO_4 and Sp_4 the complexification of the $\mathbb{Z}$–module $X(T)$ of k–rational characters of T is denoted by $\mathfrak{a}^*_{T,\mathbb{C}}$. It is two–dimensional and we identify it with $\mathbb{C}^2$ taking for the basis reduced norms on every copy of GL_1 inside T.

We should remark that in this paper the usual parabolic induction from standard parabolic subgroup P of G with the Levi factor M will be denoted by Ind_M^G instead of Ind_P^G. This will not cause any confusion since all the parabolic subgroups appearing in the paper are standard.

2. Normalizing factors for intertwining operators

In this Section we recall results on normalization of intertwining operators obtained in [6] and [7]. The global normalizing factors are given for standard global intertwining operators in all maximal parabolic subgroup cases needed in the sequel. These factors are scalar valued meromorphic functions with the property that the normalized intertwining operator becomes holomorphic and non–vanishing in the closure of the positive Weyl chamber, except at the origin in some cases. We do not repeat the proof of holomorphy and non–vanishing in the cases needed in this paper since it can be found in [6] and [7]. The strategy of the proof is local. At split places the normalization is obtained using Langlands–Shahidi method explained in [25] and [26]. Non–split places for non–quasi split groups are out of scope of Langlands–Shahidi method and the normalization is obtained using a new technique developed in [6] and [7] which is based on transfer of Plancherel measures between split groups and their inner forms as in [24].

The normalizing factors are defined using Jacquet–Langlands correspondence explained in Section 8 of [5]. More precisely, let $\pi'\cong\otimes_v\pi_v'$ be a higher–dimensional cuspidal automorphic representation of $GL_1'(\mathbb{A})$. Then, at non–split places $v\in S$, by the local Jacquet–Langlands correspondence of Theorem (8.1) in [5], π_v' corresponds to a square–integrable representation π_v of $GL_2(k_v)$. At the split places $v\notin S$ we have $GL_1'(k_v)\cong GL_2(k_v)$ and we take $\pi_v\cong\pi_v'$. Then, by the global Jacquet–Langlands correspondence, π' corresponds to a representation $\pi\cong\otimes_v\pi_v$ of $GL_2(\mathbb{A})$. By Theorem (8.3) of [5], π is isomorphic to a cuspidal automorphic representation of $GL_2(\mathbb{A})$.

Otherwise, a cuspidal automorphic representation π' of $GL_1'(\mathbb{A})$ is one–dimensional. Then, there exists a unitary character χ of $\mathbb{A}^\times/k^\times$ such that $\pi'\cong\chi\circ\det'$.

Now, we are ready to give global normalizing factors for standard intertwining operators in maximal proper parabolic cases needed in the sequel. Those cases have Levi factors $GL_1'\subset G_1'$, $GL_1'\subset H_1'$ for non–quasi split groups and $GL_2\subset SO_4$, $GL_2\subset Sp_4$, $GL_1\times GL_1\subset GL_2$ and $GL_1\subset Sp_2\cong SL_2$ for split groups. Furthermore, for non–quasi split groups the normalizing factors depend on dimension of a cuspidal automorphic representation of $GL_1'(\mathbb{A})$, i.e. factors are different for higher–dimensional and one–dimensional representations. The normalizing factor in general proper parabolic case is just a product of normalizing factors in maximal cases appearing in decomposition of standard intertwining operator given in Section 2.1 of [25].

For a cuspidal automorphic representation σ of one of the Levi factors $M(\mathbb{A})$, $\underline{s} \in \mathfrak{a}_{M,\mathbb{C}}^*$ and the unique nontrivial Weyl group element w_0 corresponding to M, the standard intertwining operator, denoted by $A(\underline{s}, \sigma, w_0)$, acts on the induced representation $I(\underline{s}, \sigma)$ where I denotes normalized induction for the corresponding maximal proper parabolic case. Observe that $\underline{s} = s \in \mathbb{C}$ in all cases except $GL_1 \times GL_1 \subset GL_2$ when $\underline{s} = (s_1, s_2) \in \mathbb{C}^2$. The normalizing factor for the standard intertwining operator $A(\underline{s}, \sigma, w_0)$ is denoted by $r(\underline{s}, \sigma, w_0)$. Then, the normalized intertwining operator $N(\underline{s}, \sigma, w_0)$ is given by

$$A(\underline{s}, \sigma, w_0) = r(\underline{s}, \sigma, w_0) N(\underline{s}, \sigma, w_0).$$

It is proved in Section 1 of [6] and Section 1 of [7] that the normalized operators obtained in this way are holomorphic and non–vanishing in the region required in this paper. Thus, calculation of poles of standard intertwining operators is reduced to poles of normalizing factors.

When defining global normalizing factors in [6] and [7], the first step is to define local normalizing factors and local normalized intertwining operators at all places. In this paper we use local normalized intertwining operators when studying images of global ones. However, for precise definition one should consult [6] and [7].

In the case $GL_1' \subset G_1'$, the normalizing factor for the intertwining operator $A(s, \pi', w)$ acting on the induced representation

$$\mathrm{Ind}_{GL_1'(\mathbb{A})}^{G_1'(\mathbb{A})} \pi' \nu^s,$$

where π' is a higher–dimensional cuspidal automorphic representation of $GL_1'(\mathbb{A})$, is given by

$$r(s, \pi', w) = \frac{L(2s, \omega_\pi)}{L(1+2s, \omega_\pi)\varepsilon(2s, \omega_\pi)}, \tag{1}$$

where L–function and ε–factor are the ones of Hecke for the central character ω_π of a cuspidal automorphic representation π of $GL_2(\mathbb{A})$ corresponding to π' by the global Jacquet–Langlands correspondence.

In the case $GL_1' \subset G_1'$, the normalizing factor for the intertwining operator $A(s, \chi \circ \det', w)$ acting on the induced representation

$$\mathrm{Ind}_{GL_1'(\mathbb{A})}^{G_1'(\mathbb{A})} (\chi \circ \det') \nu^s,$$

where $\chi \circ \det'$ is a one–dimensional cuspidal automorphic representation of $GL_1'(\mathbb{A})$, is given by

$$r(s, \chi \circ \det', w) = \frac{L(2s, \chi^2)}{L(1+2s, \chi^2)\varepsilon(2s, \chi^2)}, \tag{2}$$

where L–function and ε–factor are the ones of Hecke.

In the case $GL_1' \subset H_1'$, the normalizing factor for the intertwining operator $A(s, \pi', w)$ acting on the induced representation

$$\mathrm{Ind}_{GL_1'(\mathbb{A})}^{H_1'(\mathbb{A})} \pi' \nu^s,$$

where π' is a higher–dimensional cuspidal automorphic representation of $GL_1'(\mathbb{A})$, is given by

$$r(s, \pi', w) = \frac{L(s, \pi)}{L(1+s, \pi)\varepsilon(s, \pi)} \cdot \frac{L(2s, \omega_\pi)}{L(1+2s, \omega_\pi)\varepsilon(2s, \omega_\pi)}, \tag{3}$$

where L–functions and ε–factors are principal Jacquet L–functions and ε–factors for π and the ones of Hecke for the central character ω_π of π. Here π is a cuspidal automorphic representation of $GL_2(\mathbb{A})$ corresponding to π' by the global Jacquet–Langlands correspondence.

In the case $GL_1' \subset H_1'$, the normalizing factor for the intertwining operator $A(s,\chi\circ\det',w)$ acting on the induced representation

$$\mathrm{Ind}_{GL_1'(\mathbb{A})}^{H_1'(\mathbb{A})}(\chi\circ\det')\nu^s,$$

where $\chi\circ\det'$ is a one–dimensional cuspidal automorphic representation of $GL_1'(\mathbb{A})$, is given by

$$r(s,\chi\circ\det',w)= \tag{4}$$

$$=\frac{L(2s,\chi^2)}{L(1+2s,\chi^2)\varepsilon(2s,\chi^2)}\cdot\frac{L(s-1/2,\chi)}{L(s+3/2,\chi)\varepsilon(s+1/2,\chi)\varepsilon(s-1/2,\chi)}\prod_{v\in S}\frac{L(s+1/2,\chi_v)}{L(1/2-s,\chi_v^{-1})},$$

where L–functions and ε–factors are the global and the local ones of Hecke.

In the case $GL_2\subset SO_4$, the normalizing factor for the intertwining operator $A(s,\pi,w)$ acting on the induced representation

$$\mathrm{Ind}_{GL_2(\mathbb{A})}^{SO_4(\mathbb{A})}\pi\nu^s,$$

where π is a cuspidal automorphic representation of $GL_2(\mathbb{A})$, is given by

$$r(s,\pi,w)=\frac{L(2s,\omega_\pi)}{L(1+2s,\omega_\pi)\varepsilon(2s,\omega_\pi)}, \tag{5}$$

where L–function and ε–factor are the ones of Hecke for the central character ω_π of π.

In the case $GL_2\subset Sp_4$, the normalizing factor for the intertwining operator $A(s,\pi,w)$ acting on the induced representation

$$\mathrm{Ind}_{GL_2(\mathbb{A})}^{Sp_4(\mathbb{A})}\pi\nu^s,$$

where π is a cuspidal automorphic representation of $GL_2(\mathbb{A})$, is given by

$$r(s,\pi,w)=\frac{L(s,\pi)}{L(1+s,\pi)\varepsilon(s,\pi)}\cdot\frac{L(2s,\omega_\pi)}{L(1+2s,\omega_\pi)\varepsilon(2s,\omega_\pi)}, \tag{6}$$

where L–functions and ε–factors are principal Jacquet L–functions and ε–factors for π and the ones of Hecke for the central character ω_π of π.

In the case $GL_1\times GL_1\subset GL_2$, the normalizing factor for the standard intertwining operator $A((s_1,s_2),\chi_1\otimes\chi_2,w_1)$ acting on the induced representation

$$\mathrm{Ind}_{GL_1(\mathbb{A})\times GL_1(\mathbb{A})}^{GL_2(\mathbb{A})}\left(\chi_1|\cdot|^{s_1}\otimes\chi_2|\cdot|^{s_2}\right),$$

where χ_1 and χ_2 are unitary characters of $\mathbb{A}^\times/k^\times$, is given by

$$r((s_1,s_2),\chi_1\otimes\chi_2,w_1)=\frac{L(s_1-s_2,\chi_1\chi_2^{-1})}{L(1+s_1-s_2,\chi_1\chi_2^{-1})\varepsilon(s_1-s_2,\chi_1\chi_2^{-1})}, \tag{7}$$

where L–functions and ε–factors are the ones of Hecke.

In the case $GL_1\subset SL_2$, the normalizing factor for the intertwining operator $A(s,\chi,w_2)$ acting on the induced representation

$$\mathrm{Ind}_{GL_1(\mathbb{A})}^{SL_2(\mathbb{A})}\chi|\cdot|^s,$$

where χ is a unitary character of $\mathbb{A}^\times/k^\times$, is given by

$$r(s,\chi,w_2)=\frac{L(s,\chi)}{L(1+s,\chi)\varepsilon(s,\chi)}, \tag{8}$$

where L–functions and ε–factors are the ones of Hecke.

Observe that for SO_4 the intertwining operator $A((s_1,s_2),\chi_1\otimes\chi_2,w_2)$, where χ_1 and χ_2 are unitary characters of $\mathbb{A}^\times/k^\times$, is in fact intertwining operator for $GL_1\times GL_1\subset GL_2$ case since the Levi factor of the parabolic subgroup corresponding to the simple root e_1+e_2 is isomorphic to GL_2. However,

$$r((s_1,s_2),\chi_1\otimes\chi_2,w_2)=r((s_1,-s_2),\chi_1\otimes\chi_2^{-1},w_1)=\frac{L(s_1+s_2,\chi_1\chi_2)}{L(1+s_1+s_2,\chi_1\chi_2)\varepsilon(s_1+s_2,\chi_1\chi_2)}. \tag{9}$$

Finally, we recall well-known analytic properties of local and global L–functions appearing in the normalizing factors above. The proofs for Hecke L–functions can be found in [28] and for principal Jacquet L–functions for GL_2 in [10]. Observe that the global Hecke L–function $L(s,\mathbf{1})$ for the trivial character $\mathbf{1}$ of $\mathbb{A}^\times/k^\times$ is nothing else than the complete ζ–function of algebraic number field k.

Lemma 2.1. *The global principal Jacquet L–function $L(s,\sigma)$ of a cuspidal automorphic representation σ of $GL_2(\mathbb{A})$ is entire. It has no zeroes for $Re(s)\geqslant 1$.*

The global Hecke L–function $L(s,\mu)$ of a unitary character μ of $\mathbb{A}^\times/k^\times$ has simple poles at $s=0$ and $s=1$ if μ is trivial and it is entire otherwise. It has no zeroes for $Re(s)\geqslant 1$. The local Hecke L–function $L(s,\mu_v)$ of a unitary character μ_v of $k_v^\times$ has a simple real pole at $s=0$ if μ_v is trivial and it is entire otherwise. It has no zeroes.

3. Calculation of the residual spectra

In this Section we decompose the residual spectra of $G_1'(\mathbb{A})$ and $H_1'(\mathbb{A})$ and parts of the residual spectra of $SO_4(\mathbb{A})$ and $Sp_4(\mathbb{A})$ required for defining the correspondence. The strategy of decomposition is Langlands spectral theory explained in [17] and [22]. Constituents of the residual spectrum are spaces of automorphic forms obtained as iterated residues inside the positive Weyl chamber of Eisenstein series attached to cuspidal automorphic representations of Levi factors of standard proper parabolic subgroups which are square–integrable. The calculation of poles reduces to the poles of the constant term of the Eisenstein series since it inherits all analytic properties from the Eisenstein series. Furthermore, the constant term equals the sum of standard intertwining operators and their poles are given by the normalizing factors of the previous Section. Square integrability of the iterated residues is checked using Langlands square integrability criterion from page 104 of [17].

During calculation of poles of Eisenstein series we always assume that they are real. There is no loss in generality because that can be achieved just by twisting a cuspidal automorphic representation of a Levi factor by the appropriate imaginary power of the absolute value of the reduced norm of the determinant. Hence, this assumption is just a convenient choice of coordinates.

Let $L^2_{res}(G_1')$ and $L^2_{res}(H_1')$ be the residual spectra of $G_1'(\mathbb{A})$ and $H_1'(\mathbb{A})$, respectively. Since those groups have only one proper parabolic subgroup P_0' with the Levi factor $M_0'\cong GL_1'$, constituents of the residual spectrum are iterated residues of the Eisenstein series attached to cuspidal automorphic representations of $M_0'(\mathbb{A})$.

Let $L^2_{res}(SO_4)$ and $L^2_{res}(Sp_4)$ be the residual spectra of $SO_4(\mathbb{A})$ and $Sp_4(\mathbb{A})$, respectively. For every standard proper parabolic subgroup P of SO_4 or Sp_4 with the Levi factor M let $L^2_{res,M}(SO_4)$ or $L^2_{res,M}(Sp_4)$ be the part of the residual spectrum which is obtained as iterated residues of the Eisenstein series attached to cuspidal automorphic representations of $M(\mathbb{A})$. Accordingly, the residual spectrum decomposes into the direct sum

$$L^2_{res}(SO_4) \cong \oplus_M L^2_{res,M}(SO_4) \quad \text{and} \quad L^2_{res}(Sp_4) \cong \oplus_M L^2_{res,M}(Sp_4).$$

However, only parts attached to Levi factors $M_0 \cong GL_2$ and $T \cong GL_1 \times GL_1$ are involved in the correspondence between the residual spectra of $G'_1(\mathbb{A})$ and $H'_1(\mathbb{A})$ and the residual spectra of $SO_4(\mathbb{A})$ and $Sp_4(\mathbb{A})$. Thus, in this paper we decompose only those parts of the residual spectra.

Theorem 3.1. *The residual spectrum $L^2_{res}(G'_1)$ of the group $G'_1(\mathbb{A})$ decomposes into the direct sum*

$$L^2_{res}(G'_1) \cong \left(\oplus_{\pi'} \mathcal{A}_{G'_1}(\pi')\right) \oplus \left(\oplus_{\chi} \mathcal{A}_{G'_1}(\chi)\right).$$

The former sum is over all higher–dimensional cuspidal automorphic representations π' of $M'_0(\mathbb{A}) \cong GL'_1(\mathbb{A})$ having trivial central character. The latter sum is over all quadratic characters χ of $\mathbb{A}^\times/k^\times$.

The irreducible space of automorphic forms $\mathcal{A}_{G'_1}(\pi')$ is spanned by the residue at $s = 1/2$ of the Eisenstein series attached to π' which is by the constant term map isomorphic to the image of the normalized intertwining operator $N(1/2, \pi', w)$.

The irreducible space of automorphic forms $\mathcal{A}_{G'_1}(\chi)$ is spanned by the residue at $s = 1/2$ of the Eisenstein series attached to the one–dimensional cuspidal automorphic representation $\chi \circ \det'$ of $M'_0(\mathbb{A}) \cong GL'_1(\mathbb{A})$ which is by the constant term map isomorphic to the image of the normalized intertwining operator $N(1/2, \chi \circ \det', w)$.

Proof. The constant term of the Eisenstein series attached to both π' and $\chi \circ \det'$ is the sum over W' of standard intertwining operators. Since w is the only nontrivial element of W', poles of the Eisenstein series inside the positive Weyl chamber $s > 0$ are those of the standard intertwining operator corresponding to w. Moreover, by the previous Section, the normalized intertwining operator is holomorphic and non–vanishing for $s > 0$ and hence poles of the standard intertwining operator for $s > 0$ coincide with poles of the normalizing factor.

For higher–dimensional π' the normalizing factor is given by (1) and by Lemma 2.1 it has a pole if and only if ω_π is trivial, where π corresponds to π' by Jacquet–Langlands correspondence. Then the simple pole is at $s = 1/2$ and its residue, up to a non–zero constant, equals $N(1/2, \pi', w)$. Observe that $\omega_\pi = \omega_{\pi'}$ because central characters are invariant for Jacquet–Langlands correspondence.

For one–dimensional $\chi \circ \det'$ the normalizing factor is given by (2) and by Lemma 2.1 it has a pole if and only if χ^2 is trivial. Then the simple pole is at $s = 1/2$ and its residue, up to a non–zero constant, equals $N(1/2, \chi \circ \det', w)$.

Langlands square–integrability criterion is obviously satisfied in both cases. It remains to prove that the images of $N(1/2, \pi', w)$ and $N(1/2, \chi \circ \det', w)$ are irreducible. That is done locally for every place v of k. If π'_v or $\chi_v \circ \det'_v$ is tempered, the image is irreducible by Langlands classification since $1/2$ is in the positive Weyl chamber. Observe that this is the case for all $v \in S$.

Let $v \notin S$ and $\pi_v \cong \pi'_v$ a non–tempered representation of $GL_2(k_v)$. Then, as a local component of a cuspidal automorphic representation of $GL_2(\mathbb{A})$, it is unitary and generic. Thus π_v is a complementary series, i.e. a fully induced representation

$$\pi_v \cong \mathrm{Ind}_{GL_1(k_v) \times GL_1(k_v)}^{GL_2(k_v)} \left(\mu_v |\cdot|^r \otimes \mu_v |\cdot|^{-r}\right),$$

where $0 < r < 1/2$ and μ_v is a unitary character of $k_v^\times$. Then, the image of $N(1/2, \pi'_v, w)$ is the same as the image of $N((1/2+r, 1/2-r), \mu_v \otimes \mu_v, w_1w_2)$ which is irreducible by the Langlands classification since $(1/2+r, 1/2-r)$ is in the positive Weyl chamber for $T \subset SO_4$ and w_1w_2 the longest Weyl group element.

Similarly, $\chi_v \circ \det_v$ at $v \notin S$ is the Langlands quotient of the standard module

$$\mathrm{Ind}_{GL_1(k_v)\times GL_1(k_v)}^{GL_2(k_v)} \left(\chi_v|\cdot|^{1/2} \otimes \chi_v|\cdot|^{-1/2}\right).$$

In other words it is the image of the normalized intertwining operator $N((1/2, -1/2), \chi_v \otimes \chi_v, w_1)$. Hence, the image of $N(1/2, \chi_v \circ \det'_v, w)$ is the same as the image of $N((1, 0), \chi_v \otimes \chi_v, w_1w_2)$ which is irreducible by Langlands classification as above. $\square$

Before decomposing $L^2_{res}(H'_1)$ we introduce some notation. Let $\chi \circ \det' = \otimes_v(\chi_v \circ \det'_v)$ be a one–dimensional cuspidal automorphic representation of the Levi factor $GL'_1(\mathbb{A})$ in $H'_1(\mathbb{A})$ where χ is a quadratic character of $\mathbb{A}^\times/k^\times$. For a non–split place $v \in S$ let Π'_v denote the image of the local normalized intertwining operator $N(1/2, \chi_v \circ \det'_v, w)$. Since $\chi_v \circ \det'_v$ is supercuspidal and $1/2$ is in the open positive Weyl chamber, that image is irreducible by the Langlands classification.

For a split place $v \notin S$ consider the image of the normalized operator $N(1/2, \chi_v \circ \det'_v, w)$. It is isomorphic to the image of the normalized intertwining operator $N((1, 0), \chi_v \otimes \chi_v, w_1w_2w_1w_2)$ because $w = w_2w_1w_2$ and $\chi_v \circ \det'$ is the image of the $GL_2(k_v)$ normalized intertwining operator $N((1/2, -1/2), \chi_v \otimes \chi_v, w_1)$. By decomposition property of Section 2.1 in [25], normalized operator

$$N((1, 0), \chi_v \otimes \chi_v, w_1w_2w_1w_2) = N((1, 0), \chi_v \otimes \chi_v, w_1w_2w_1)N((1, 0), \chi_v \otimes \chi_v, w_2).$$

By inducing in stages we have

$$\mathrm{Ind}_{T(k_v)}^{Sp_4(k_v)} (\chi_v|\cdot| \otimes \chi_v) \cong \mathrm{Ind}_{GL_1(k_v)\times SL_2(k_v)}^{Sp_4(k_v)} \left(\chi_v|\cdot| \otimes \mathrm{Ind}_{GL_1(k_v)}^{SL_2(k_v)}\chi_v\right).$$

Observe that $N((1, 0), \chi_v \otimes \chi_v, w_2)$ intertwines the $SL_2(k_v)$ induced representation with itself. Furthermore,

$$\mathrm{Ind}_{GL_1(k_v)}^{SL_2(k_v)}\chi_v \cong \tau_v^+ \oplus \tau_v^-$$

where $\tau_v^\pm$ are irreducible tempered representations of $SL_2(k_v)$ and the sign in the exponent denotes the sign of action of $SL_2(k_v)$ normalized intertwining operator $N((1, 0), \chi_v \otimes \chi_v, w_2)$. Moreover, $\tau_v^\pm$ are both nontrivial unless χ_v is trivial and then τ_v^- is trivial. If χ_v is unramified, then the unramified component of the induced representation is τ_v^+ since, by definition, the normalized operator acts as identity on the unramified component. Now, $w_1w_2w_1$ is the longest Weyl group element for the standard proper parabolic subgroup of Sp_4 with Levi factor isomorphic to $GL_1 \times SL_2$, 1 is in the positive Weyl chamber and $\chi_v \otimes \tau_v^\pm$ is tempered. Therefore, the image of the operator $N((1, 0), \chi_v \otimes \chi_v, w_1w_2w_1)$ acting on

$$\mathrm{Ind}_{GL_1(k_v)\times SL_2(k_v)}^{Sp_4(k_v)} \left(\chi_v|\cdot| \otimes \tau_v^\pm\right)$$

is irreducible by Langlands classification and we denote it by $\Pi'^\pm_v$. Then the image of the normalized intertwining operator $N(1/2, \chi_v \circ \det'_v, w)$ decomposes into the direct sum

$$\Pi'^+_v \oplus \Pi'^-_v,$$

where Π'^-_v is trivial if χ_v is trivial. Observe that representations Π'_v at $v \in S$ and $\Pi'^\pm_v$ at $v \notin S$ depend on the quadratic character χ_v although it is not explicit in our notation. Moreover, since τ_v^+ is the unramified component for unramified χ_v, Π'^-_v is never unramified.

Theorem 3.2. *The residual spectrum $L^2_{res}(H'_1)$ of the group $H'_1(\mathbb{A})$ decomposes into the direct sum*

$$L^2_{res}(H'_1) \cong \Big(\oplus_{\pi'} \mathcal{A}_{H'_1}(\pi')\Big) \oplus \Big(\oplus_{\chi} \mathcal{A}_{H'_1}(\chi)\Big) \oplus \mathcal{A}_{H'_1}(\mathbf{1}).$$

The former sum is over all higher–dimensional cuspidal automorphic representations π' of $M'_0(\mathbb{A}) \cong GL'_1(\mathbb{A})$ which have trivial central character and $L(1/2,\pi) \neq 0$ for the global principal Jacquet L–function of a cuspidal automorphic representation π corresponding to π' by the Jacquet–Langlands correspondence. The latter sum is over all quadratic characters χ of $\mathbb{A}^\times/k^\times$ such that χ_v is non-trivial at all non–split places $v \in S$. Finally, $\mathbf{1}$ denotes the trivial character of $\mathbb{A}^\times/k^\times$.

The irreducible space of automorphic forms $\mathcal{A}_{H'_1}(\pi')$ is spanned by the residue at $s = 1/2$ of the Eisenstein series attached to π' which is by the constant term map isomorphic to the image of the normalized intertwining operator $N(1/2,\pi',w)$.

The space of automorphic forms $\mathcal{A}_{H'_1}(\chi)$, for nontrivial χ, is spanned by the residue at $s = 1/2$ of the Eisenstein series attached to the one–dimensional cuspidal automorphic representation $\chi \circ \det'$ of $M'_0(\mathbb{A}) \cong GL'_1(\mathbb{A})$ which is by the constant term map isomorphic to the image of the normalized intertwining operator $N(1/2, \chi \circ \det', w)$. It decomposes into the sum of irreducible constituents which are by the constant term map isomorphic to irreducible representations of $H'_1(\mathbb{A})$ of the form $\otimes_v \Pi'_v$, where Π'_v at a non–split place $v \in S$ is defined above, while at a split place $v \notin S$ it is one of representations $\Pi'^{\pm}_v$ defined above and it is Π'^{+}_v at almost all places.

The irreducible space of automorphic forms $\mathcal{A}_{H'_1}(\mathbf{1})$ is spanned by the iterated residue at $s = 3/2$ of the Eisenstein series attached to $\mathbf{1} \circ \det'$ which is by the constant term map isomorphic to the image of the normalized intertwining operator $N(3/2, \mathbf{1} \circ \det', w)$.

Proof. The proof goes along the same lines as the proof of the previous Theorem 3.1. Poles of the Eisenstein series inside the positive Weyl chamber $s > 0$ coincide with poles of the normalizing factors for the standard intertwining operator corresponding to w.

For higher–dimensional π' the normalizing factor is given by (3) and by Lemma 2.1 it has a pole if and only if ω_π is trivial and $L(1/2,\pi) \neq 0$, where π corresponds to π' by Jacquet–Langlands correspondence. Then the simple pole is at $s = 1/2$ and its residue, up to a non–zero constant, equals $N(1/2,\pi',w)$. Observe that $\omega_\pi = \omega_{\pi'}$ because central characters are invariant for Jacquet–Langlands correspondence.

For one–dimensional $\chi \circ \det'$ the normalizing factor is given by (4) and by Lemma 2.1 its possible poles are at $s = 1/2$ and $s = 3/2$. The pole at $s = 1/2$ occurs if and only if χ^2 is trivial but χ_v is nontrivial at all non–split places $v \in S$. It is simple and the residue, up to a non–zero constant, equals $N(1/2, \chi \circ \det', w)$. The local condition on χ_v appears since otherwise the pole of local L–functions $L(1/2 - s, \chi_v^{-1})$ in denominator of (4) would cancel the pole of numerator. The pole at $s = 3/2$ occurs if and only if χ is trivial. It is simple and the residue, up to a nonzero constant, equals $N(3/2, \mathbf{1} \circ \det', w)$.

Langlands square–integrability criterion is obviously satisfied in both cases. Irreducibility of images of $N(1/2,\pi',w)$ and $N(3/2, \mathbf{1} \circ \det', w)$ follows as in the proof of the previous Theorem 3.1. Moreover, the same argument shows that the image of the local normalized intertwining operator $N(1/2, \chi_v \circ \det_v, w)$ at a split place $v \in S$ is the same as the image of $N((1,0), \chi_v \otimes \chi_v, w_1 w_2 w_1 w_2)$. By the discussion preceding the statement of this Theorem, that image is the sum of Π'^{+}_v and Π'^{-}_v. At a non–split place $v \in S$ the image of the local normalized operator $N(1/2, \chi_v \circ \det'_v, w)$ is irreducible by Langlands classification and denoted by Π'_v above. From images of the local normalized intertwining operators at all places we obtain decomposition of the image of the global normalized intertwining operator $N(1/2, \chi \circ \det', w)$. At almost all places Π'_v is Π'^{+}_v because Π'^{-}_v is never an unramified representation of $Sp_4(k_v)$ as explained above. $\square$

Theorem 3.3. *The part $L^2_{res,M_0}(SO_4)$ of the residual spectrum $L^2_{res}(SO_4)$ of the group $SO_4(\mathbb{A})$ decomposes into the direct sum*

$$L^2_{res,M_0}(SO_4) \cong \oplus_\pi \mathcal{A}^{M_0}_{SO_4}(\pi),$$

where the sum is over all cuspidal automorphic representations π of $M_0(\mathbb{A}) \cong GL_2(\mathbb{A})$ having trivial central character. The irreducible space of automorphic forms $\mathcal{A}^{M_0}_{SO_4}(\pi)$ is spanned by the iterated residue at $s = 1/2$ of the Eisenstein series attached to π which is by the constant term map isomorphic to the image of the normalized intertwining operator $N(1/2, \pi, w)$.

The part $L^2_{res,T}(SO_4)$ of the residual spectrum $L^2_{res}(SO_4)$ of the group $SO_4(\mathbb{A})$ decomposes into the direct sum

$$L^2_{res,T}(SO_4) \cong \oplus_\chi \mathcal{A}^{T}_{SO_4}(\chi),$$

where the sum is over all quadratic characters χ of $\mathbb{A}^\times/k^\times$. The irreducible space of automorphic forms $\mathcal{A}^{T}_{SO_4}(\chi)$ is spanned by the iterated residue at $(s_1, s_2) = (1, 0)$ of the Eisenstein series attached to cuspidal automorphic representation $\chi \otimes \chi$ of $T(\mathbb{A}) \cong GL_1(\mathbb{A}) \times GL_1(\mathbb{A})$ which is by the constant term map isomorphic to the image of the normalized intertwining operator $N((1, 0), \chi \otimes \chi, w_1w_2)$.

Proof. First we decompose the part $L^2_{res,M_0}(SO_4)$. Since the only nontrivial element of $W(M_0)$ is w, poles inside the positive Weyl chamber $s > 0$ of the Eisenstein series attached to a cuspidal automorphic representation π of $M_0(\mathbb{A})$ coincide with poles of the normalizing factor for the standard intertwining operator corresponding to w. It is given by (5) and by Lemma 2.1 the pole occurs if and only if the central character ω_π is trivial. Then, the pole is at $s = 1/2$, it is simple and its residue, up to a non–zero constant, equals $N(1/2, \pi, w)$. The Langlands square–integrability criterion is obviously satisfied and the image of that operator is irreducible by the same argument as in the proof of Theorem 3.1.

For the part $L^2_{res,T}(SO_4)$, the constant term of the Eisenstein series attached to a cuspidal automorphic representation $\chi_1 \otimes \chi_2$ of $T(\mathbb{A})$ is the sum over $W = \{1, w_1, w_2, w_1w_2\}$ of the standard intertwining operators. Normalizing factors corresponding to w_1 and w_2 are given by (7) and (9) and the normalizing factor corresponding to w_1w_2 is just the product of those two. Hence, by Lemma 2.1, possible poles inside the positive Weyl chamber $s_1 > |s_2|$ of the normalizing factors are along hyperplanes $s_1 - s_2 = 1$ and $s_1 + s_2 = 1$. The hyperplane $s_1 - s_2 = 1$ is singular if and only if $\chi_1 = \chi_2$ and hyperplane $s_1 + s_2 = 1$ is singular if and only if $\chi_1 = \chi_2^{-1}$. Then the pole along both is simple. Therefore, the only possible iterated pole is at their intersection, i.e. for $(s_1, s_2) = (1, 0)$ and the Weyl group element w_1w_2. It occurs if and only if $\chi = \chi_1 = \chi_2$ is a quadratic character and its residue, up to a non–zero constant, equals $N((1, 0), \chi \otimes \chi, w_1w_2)$. That image is irreducible by the Langlands classification because $(1, 0)$ is in the positive Weyl chamber and w_1w_2 is the longest Weyl group element. The Langlands square integrability criterion is satisfied since $w_1w_2(1, 0) = (-1, 0)$. $\square$

The residual spectrum for the remaining case of the split group $Sp_4(\mathbb{A})$ is decomposed by H. Kim in [11]. For convenience, we state his Theorem 3.3 and Theorem 5.4 in our notation in the following Theorem 3.4 below. Recall that before the statement of Theorem 3.2 we decomposed the image of the local normalized intertwining operator $N((1, 0), \chi_v \otimes \chi_v, w_1w_2w_1w_2)$ into the sum

$$\Pi^+_v \oplus \Pi^-_v,$$

where Π_v^- is trivial if χ_v is trivial.

Theorem 3.4 (Kim, [11]). *The part $L^2_{res,M_0}(Sp_4)$ of the residual spectrum $L^2_{res}(Sp_4)$ of the group $Sp_4(\mathbb{A})$ decomposes into the direct sum*

$$L^2_{res,M_0}(Sp_4) \cong \oplus_\pi \mathcal{A}^{M_0}_{Sp_4}(\pi),$$

where the sum is over all cuspidal automorphic representations π of $M_0(\mathbb{A}) \cong GL_2(\mathbb{A})$ having trivial central character and such that $L(1/2,\pi) \neq 0$. The irreducible space of automorphic forms $\mathcal{A}^{M_0}_{Sp_4}(\pi)$ is spanned by the iterated residue at $s = 1/2$ of the Eisenstein series attached to π which is by the constant term map isomorphic to the image of the normalized intertwining operator $N(1/2,\pi,w)$.

The part $L^2_{res,T}(Sp_4)$ of the residual spectrum $L^2_{res}(Sp_4)$ of the group $Sp_4(\mathbb{A})$ decomposes into the direct sum

$$L^2_{res,T}(Sp_4) \cong \left(\oplus_\chi \mathcal{A}^T_{Sp_4}(\chi)\right) \oplus \mathcal{A}^T_{Sp_4}(\mathbf{1}),$$

where the sum is over all nontrivial quadratic characters χ of $\mathbb{A}^\times/k^\times$ and $\mathbf{1}$ is the trivial character of $\mathbb{A}^\times/k^\times$.

The space of automorphic forms $\mathcal{A}^T_{Sp_4}(\chi)$, for nontrivial χ, is spanned by the iterated residue at $(s_1,s_2) = (1,0)$ of the Eisenstein series attached to cuspidal automorphic representation $\chi \otimes \chi$ of $T(\mathbb{A}) \cong GL_1(\mathbb{A}) \times GL_1(\mathbb{A})$. It decomposes into the sum of irreducible constituents which are by the constant term map isomorphic to irreducible representations of $Sp_4(\mathbb{A})$ of the form $\otimes_v \Pi_v$, where Π_v is one of representations $\Pi_v^\pm$, it is Π_v^+ at almost all places and the product of the signs over all places equals 1.

The irreducible space of automorphic forms $\mathcal{A}^T_{Sp_4}(\mathbf{1})$ is spanned by the iterated residue at $(s_1,s_2) = (2,1)$ of the Eisenstein series attached to the trivial cuspidal automorphic representation $\mathbf{1} \otimes \mathbf{1}$ of $T(\mathbb{A}) \cong GL_1(\mathbb{A}) \times GL_1(\mathbb{A})$ which is by the constant term map isomorphic to the image of the normalized intertwining operator $N((2,1),\mathbf{1}\otimes\mathbf{1},w_1w_2w_1w_2)$.

Finally, we obtain correspondence between the residual spectra of $G'_1(\mathbb{A})$ and $SO_4(\mathbb{A})$ and between the residual spectra of $H'_1(\mathbb{A})$ and $Sp_4(\mathbb{A})$.

Corollary 3.5. *Mapping $\imath$ defined, in notation of Theorem 3.1 and Theorem 3.3, by*

$$\begin{aligned} \imath\left(\mathcal{A}_{G'_1}(\pi')\right) &= \mathcal{A}^{M_0}_{SO_4}(\pi), \\ \imath\left(\mathcal{A}_{G'_1}(\chi)\right) &= \mathcal{A}^{T}_{SO_4}(\chi), \end{aligned}$$

where π corresponds to π' by the Jacquet–Langlands correspondence, is an injective map from irreducible constituents of $L^2_{res}(G'_1)$ to irreducible constituents of $L^2_{res,M_0}(SO_4) \oplus L^2_{res,T}(SO_4)$. The image of the map $\imath$ consists of

(a) *all irreducible constituents $\mathcal{A}^{M_0}_{SO_4}(\pi)$ of $L^2_{res,M_0}(SO_4)$ such that π_v is square–integrable at every place $v \in S$, and*
(b) *all irreducible constituents $\mathcal{A}^T_{SO_4}(\chi)$ of $L^2_{res,T}(SO_4)$.*

Proof. Corollary is a direct consequence of decompositions in Theorem 3.1 and Theorem 3.3. Local square integrability condition in part (a) of the image of $\imath$ comes from the global Jacquet–Langlands correspondence. Namely, by Theorem (8.3) of [5], it is a bijection between all higher–dimensional cuspidal automorphic representations of $GL'_1(\mathbb{A})$ and all cuspidal automorphic representations of $GL_2(\mathbb{A})$ such that the local component at every place $v \in S$ is square–integrable. □

Corollary 3.6. *Mapping $\jmath$ is defined, in notation of Theorem 3.2 and Theorem 3.4, by*

$$\begin{aligned}\jmath\left(\mathcal{A}_{H_1'}(\pi')\right) &= \mathcal{A}_{Sp_4}^{M_0}(\pi),\\ \jmath\left(\otimes_v \Pi_v'\right) &= \oplus\left(\otimes_v \Pi_v\right),\\ \jmath\left(\mathcal{A}_{H_1'}(\mathbf{1})\right) &= \mathcal{A}_{Sp_4}^{T}(\mathbf{1}),\end{aligned}$$

where π corresponds to π' by the Jacquet–Langlands correspondence. In the second row $\otimes_v \Pi_v'$ is an irreducible constituent of $\mathcal{A}_{H_1'}(\chi)$ and the sum is over $2^{|S|-1}$ irreducible constituents $\otimes_v \Pi_v$ of $\mathcal{A}_{Sp_4}^T(\chi)$ such that $\Pi_v \cong \Pi_v'$ for $v \notin S$, Π_v is one of $\Pi_v^{\pm}$ for $v \in S$ and the product of all signs equals 1. Then $\jmath$ is an injective map from irreducible constituents of $L^2_{res}(H_1')$ to not necessarily irreducible constituents of $L^2_{res,M_0}(Sp_4) \oplus L^2_{res,T}(Sp_4)$. The image of the map $\jmath$ consists of

(a) *all irreducible constituents $\mathcal{A}_{Sp_4}^{M_0}(\pi)$ of $L^2_{res,M_0}(Sp_4)$ such that π_v is square–integrable at every place $v \in S$,*
(b) *all constituents of the form $\oplus(\otimes_v \Pi_v)$ as above of $\mathcal{A}_{Sp_4}^T(\chi)$ in $L^2_{res,T}(Sp_4)$ such that χ_v is nontrivial at all $v \in S$, and*
(c) *the irreducible constituent $\mathcal{A}_{Sp_4}^T(\mathbf{1})$ of $L^2_{res,T}(Sp_4)$.*

Observe that the sum of all the constituents in (b) *gives the whole space $\mathcal{A}_{Sp_4}^T(\chi)$.*

Proof. Corollary is a direct consequence of decompositions in Theorem 3.2 and Theorem 3.4. Local square integrability condition in part (a) of the image of $\jmath$ comes from the global Jacquet–Langlands correspondence as in the previous Corollary 3.5. Local non–triviality condition in part (b) comes from Theorem 3.2. The reason lies in different form of local normalizing factors for standard intertwining operators at split and non–split places resulting with local Hecke L–functions appearing in the global normalizing factor (4).

Observe that, for nontrivial χ, in decomposition of spaces $\mathcal{A}_{H_1'}(\chi)$ and $\mathcal{A}_{Sp_4}^T(\chi)$ given in Theorem 3.2 and Theorem 3.4 into irreducible constituents, the choice of local components at a split place is exactly the same. At a non–split place there is one choice for the local component of an irreducible constituent of $\mathcal{A}_{H_1'}(\chi)$ and two choices for the local component of an irreducible constituent of $\mathcal{A}_{Sp_4}^T(\chi)$. However, the parity condition for irreducible constituents of $\mathcal{A}_{Sp_4}^T(\chi)$ reduces the freedom of choice. Therefore, $\jmath$ sends irreducible constituents of $\mathcal{A}_{H_1'}(\chi)$ to the sum of $2^{|S|-1}$ irreducible components of $\mathcal{A}_{Sp_4}^T(\chi)$ as defined in the Corollary. □

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