ON A RELATION BETWEEN RESIDUAL SPECTRA OF SPLIT CLASSICAL GROUPS AND THEIR INNER FORMS

NEVEN GRBAC

Introduction

This paper is concerned with the residual spectrum of the hermitian quaternionic classical groups G'_n and H'_n defined as algebraic groups for a quaternion algebra over an algebraic number field in Section 1. Groups G'_n and H'_n are not quasi–split. They are inner forms of the split groups SO_{4n} and Sp_{4n}. Hence, the parts of the residual spectrum of G'_n and H'_n obtained in this paper are compared to the corresponding parts for the split groups SO_{4n} and Sp_{4n}. The problem of comparing the residual spectra of split groups and their inner forms is still open even for the general linear group as mentioned in [3] and Section 25 of [2].

For quasi–split groups there are many papers regarding the residual spectrum. Among them are the papers by Mœglin and Walspurger [24], Mœglin [21], [22], [23], Kim [14], [15], [18], Žampera [40], Kon–No [19]. For quasi–split groups in those papers the Langlands–Shahidi method described in [31] and [32] gives the normalization of the intertwining operators by L–functions required in the application of the Langlands spectral theory explained in [20] and [25].

Although, in principle, the results of this paper could be obtained using the Arthur trace formula explained in [2], the strategy of this paper is a more direct approach of the Langlands spectral theory and the Arthur trace formula is not used at all. However, the groups G'_n and H'_n considered in this paper are not quasi–split. Hence, they are out of the reach of the Langlands–Shahidi method and we had to develop a new technique in order to define the normalization of the intertwining operators and prove the required holomorphy and non–vanishing of the normalized intertwining operators. It is based on the lift of representations defined using the Jacquet–Langlands correspondence of [7] keeping the Plancherel measure invariant. This technique, as well as the first calculation of the residual spectrum for a non–quasi–split group, was used in the author's paper [8] where the principal series part of the residual spectrum for the group G'_2 of the semi–simple rank 2 was constructed. The invariance of the Plancherel measure was used for the first time by Muić and Savin in [30] to obtain the complementary series coming from a supercuspidal representation of the Levi factor of the Siegel parabolic subgroup for the local p–adic G'_n and H'_n. Their global idea for transferring the Plancherel measures between the split groups and their inner forms does not work for inner forms of the split groups SO_{4n+2} and Sp_{4n+2} and that is the reason why this paper restricts its attention only to inner forms of SO_{4n} and Sp_{4n}.

The main results on the residual spectrum of G'_n and H'_n are obtained in Theorem 2.2 and its Corollary 2.3 of this paper. They show certain ambiguities of quaternionic groups such as the condition on the non–triviality of the local components at all non–split places in case (ii) for the group H'_n. The reason for that lies in a different form of the local normalization factors at split and non–quasi–split places. The comparison of the parts of the residual spectrum obtained in Theorem

2.2 and the corresponding parts of the residual spectrum for split groups SO_{4n} and Sp_{4n} is given in Theorem 2.4 and Corollary 2.5.

A simple consequence of Theorem 2.2 is Corollary 2.7 showing the unitarizability of the duals under the Aubert–Schneider–Stuhler involution defined in [4] and [34] of the principal series Steinberg representations of hermitian quaternionic classical groups H'_n and G'_n defined for a quaternion algebra over a local field of characteristic zero. Namely, these duals are the local components of automorphic representations belonging to the residual spectrum obtained in Theorem 2.2. This idea of solving the unitarizability question for local representations using the fact that they are the local components of an automorphic representation belonging to the residual spectrum was used for the first time by Speh in [33] for archimedean fields and by Tadić in [35] for non–archimedean fields.

The paper is divided into two Sections. In Section 1 the normalization factors of the intertwining operators are defined and the required holomorphy and non–vanishing of the normalized intertwining operators is proved. This is done first for the local intertwining operators at a split place for generic and non–generic representations in Subsections 1.1 and 1.2 and at a non–split place in Subsection 1.3. Finally, the global normalization factors are obtained as the products of the local ones in Subsection 1.4.

Section 2 is devoted to the construction of the certain parts of the residual spectrum of the groups G'_n and H'_n coming from the minimal parabolic subgroup. The main results are Theorem 2.2 and its Corollary 2.3, as well as the comparison with the parts of the residual spectrum for split SO_{4n} and Sp_{4n} in Theorem 2.4 and Corollary 2.5. The unitarizability of the Aubert–Schneider–Stuhler duals of the principal series Steinberg representations of the local G'_n and H'_n is obtained in Corollary 2.7.

During the calculation of the poles of the Eisenstein series we always assume that they are real. There is no loss in generality because that can be achieved just by twisting a cuspidal automorphic representation of a Levi factor by the appropriate imaginary power of the absolute value of the reduced norm of the determinant. Hence, this assumption is just a convenient choice of coordinates.

We should remark that in this paper the usual parabolic induction from a standard parabolic subgroup P of G with the Levi decomposition $P = MN$ will be denoted by Ind_M^G instead of Ind_P^G. This will not cause any confusion since all the parabolic subgroups appearing in the paper are standard.

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1. Normalization of intertwining operators

Throughout this paper let k be an algebraic number field, k_v its completion at a place v and $\mathbb{A}$ its ring of adeles. Let D be a quaternion algebra central over k and τ the usual involution fixing the center of D. Then D splits at all but finitely many places v of k, i.e. at those places the completion $D \otimes_k k_v$ is isomorphic to the additive group $M(2, k_v)$ of 2×2 matrices with coefficients in k_v. At finitely many places v of k where D is non–split the completion $D \otimes_k k_v$ is isomorphic to the quaternion algebra D_v central over k_v. The finite set of places of k where D is non–split is denoted by S. The cardinality of S, denoted by $|S|$, is even for every D.

The group of invertible elements of D regarded as an algebraic group over k is denoted GL_1'. At a split place $v \notin S$ it is isomorphic to $GL_1'(k_v) \cong GL_2(k_v)$, where GL_2 is the split group of invertible 2×2 matrices. At a non–split place $v \in S$ it is isomorphic to $GL_1'(k_v) \cong D_v^\times$.

Let $\det'$ denote the reduced norm of the simple algebra $D \otimes_k \mathbb{A}$ and $\det_v'$ the corresponding reduced norm at a place v. If $v \notin S$ is split, then $\det_v' = \det_v$ is just the determinant for 2×2 matrices, while if $v \in S$ is non–split, then $\det_v'$ is the reduced norm of the quaternion algebra D_v. The absolute value of the reduced norm $\det'$ and $\det_v'$ is denoted by ν.

Let V be a $2n$–dimensional right vector space over D. We fix the basis $\{e_1, \ldots, e_{2n}\}$ of V. Then $(e_i, e_j) = \delta_{i,2n-j+1}$ for $1 \leqslant i \leqslant j \leqslant n$ defines a hermitian form on V by

$$(v, v') = \epsilon\tau((v', v)) \quad \text{and} \quad (vx, v'x') = \tau(x)(v, v')x'$$

for all $v, v' \in V$ and $x, x' \in D$, where $\epsilon \in \{\pm 1\}$. The group of isometries of the form $(\cdot, \cdot)$ regarded as a reductive algebraic group defined over k will be denoted by G_n' if $\epsilon = -1$ and by H_n' if $\epsilon = 1$. Then, G_n' is an inner form of the split group SO_{4n}, while H_n' is an inner form of the split group Sp_{4n}. Hence $G_n'(k_v) \cong SO_{4n}(k_v)$ and $H_n'(k_v) \cong Sp_{4n}(k_v)$ for every place $v \notin S$.

The maximal split torus over k for both, G_n' and H_n', is isomorphic to $GL_1 \times \ldots \times GL_1$ with n copies of GL_1. The minimal parabolic subgroup P_0' defined over k of both, G_n' and H_n', has the Levi factor $M_0' \cong GL_1' \times \ldots \times GL_1'$ with n copies of GL_1'.

The Weyl groups W' for G_n' and H_n' with respect to the maximal split torus are the same. For the corresponding split case $M_0 \cong GL_2 \times \ldots \times GL_2$ in SO_{4n} or Sp_{4n} let $W(M_0)$ denote the subgroup of the Weyl group W consisting of elements fixing the Levi factor M_0. Then $W' \cong W(M_0)$ and we will use just the symbol W' in the sequel.

Let $\mathfrak{a}_\mathbb{C}^* \cong X(M_0') \otimes_\mathbb{Z} \mathbb{C}$ denote the complexification of the $\mathbb{Z}$–module $X(M_0')$ of k–rational characters of M_0'. We fix the basis of $\mathfrak{a}_\mathbb{C}^*$ consisting of the reduced norms for every copy of GL_1'. Hence, $\mathfrak{a}_\mathbb{C}^*$ is an n–dimensional complex vector space and in the fixed basis we denote its elements as $\underline{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$. In the split case of M_0 in SO_{4n} and Sp_{4n} the space $\mathfrak{a}_\mathbb{C}^*$ is the same.

Before proceeding to the normalization we define the local and global lift of representations from GL_1' to the split GL_2. It is given by the Jacquet–Langlands correspondence explained in Section 8 of [7]. More precisely, let $\sigma' \cong \otimes_v \sigma_v'$ be a cuspidal automorphic representation of $GL_1'(\mathbb{A})$ which is not one–dimensional. Then, at non–split places $v \in S$ the local lift σ_v of σ_v' is the square–integrable representation of $GL_2(k_v)$ defined by the character relation as in Theorem (8.1) of [7]. At split places $v \notin S$ we have $GL_1'(k_v) \cong GL_2(k_v)$ and the local lift is just $\sigma_v \cong \sigma_v'$. The global lift of σ' is defined using the local lifts as $\sigma \cong \otimes_v \sigma_v$. By Theorem (8.3) of [7] the global lift σ is isomorphic to a cuspidal automorphic representation of $GL_2(\mathbb{A})$. Hence, its local components σ_v are generic.

Let $\chi \circ \det' = \otimes_v \left(\chi_v \circ \det_v'\right)$ be an one–dimensional cuspidal automorphic representation of $GL_1'(\mathbb{A})$. Here χ_v are unitary characters of $k_v^\times$ and χ is a unitary character of $\mathbb{A}^\times / k^\times$. Then the global lift of $\chi \circ \det'$ is just the one–dimensional representation $\chi \circ \det = \otimes_v (\chi_v \circ \det_v)$ of $GL_2(\mathbb{A})$. It belongs to the residual spectrum of $GL_2(\mathbb{A})$. At a non–split place $v \in S$ the local lift of $\chi_v \circ \det_v'$ is defined by the Jacquet–Langlands correspondence as in Theorem (8.1) of [7] to be the Steinberg representation of $GL_2(k_v)$ twisted by χ_v, i.e. the unique irreducible subrepresentation of the induced representation

$$\mathrm{Ind}_{GL_1(k_v)\times GL_1(k_v)}^{GL_2(k_v)}\left(\chi_v|\cdot|^{1/2}\otimes\chi_v|\cdot|^{-1/2}\right),$$

where $|\cdot|$ is the absolute value on $GL_1(k_v)\cong k_v^\times$. We denote this representation by St_{χ_v}. Observe that by our definition in this case the global and local lifts are not consistent. The reason is that the global lift is supposed to be in the discrete spectrum of $GL_2(\mathbb{A})$, while the local lift should preserve the Plancherel measure.

This Section is devoted to the definition of the scalar normalization factors for the standard intertwining operators appearing in the constant term of the Eisenstein series attached to a cuspidal automorphic representation of the Levi factor $M_0'(\mathbb{A})$ of the minimal standard parabolic subgroup P_0' of G_n' and H_n'. The main requirement of the normalized intertwining operators is to be holomorphic and non–vanishing 'deep enough' in the positive Weyl chamber so that the poles of the standard intertwining operators are captured in the normalizing factors. The expression 'deep enough' is made precise in the statements of the results below.

The normalization is obtained separately for the local normalized intertwining operators at every place v. We distinguish three cases: a split place where the representation is generic, a split place where the representation is not generic and a non–split place. Every case is treated in a separate Subsection below.

1.1. **Generic split case.** Let v be a place of k where D splits, i.e. $v\notin S$, and let G be a classical split group defined over k_v. Fix the set of positive and simple roots of G and a nontrivial continuous additive character ψ_v of k_v. For a ψ_v–generic representation of the Levi factor $M(k_v)$ of a parabolic subgroup P of G the normalization factor of the standard intertwining operators is obtained using the Langlands–Shahidi method in [32]. We recall the definition here for the convenience of the reader. For more detailed exposition see the original paper [32] or consult Section 1.1 of [8].

First, let P be the maximal proper parabolic subgroup corresponding to the subset of the set of simple roots obtained by removing the simple root α. For a ψ_v–generic representation π_v of its Levi factor $M(k_v)$ and the unique nontrivial element w of the Weyl group such that its action on the set of simple roots keeps simple all the roots in the subset defining P, the normalization factor of the standard intertwining operator

$$A(s\widetilde{\alpha},\pi_v,w)$$

acting on the induced representation

$$I(s\widetilde{\alpha},\pi_v)\cong\mathrm{Ind}_{M(k_v)}^{G(k_v)}\left(\pi_v\otimes|s\widetilde{\alpha}(\cdot)|\right),$$

is defined to be

$$r(s\widetilde{\alpha},\pi_v,w)=\prod_{i=1}^{\ell}\frac{L(is,\pi_v,r_i)}{L(1+is,\pi_v,r_i)\varepsilon(is,\pi_v,r_i,\psi_v)} \tag{1}$$

for $s\in\mathbb{C}$. Here,

$$\widetilde{\alpha}=\langle\rho_P,\alpha^\vee\rangle^{-1}\rho_P,$$

where ρ_P equals the half of the sum of the positive roots of G not being the roots of M and we write $s\widetilde{\alpha}=\widetilde{\alpha}\otimes s\in\mathfrak{a}_{M,\mathbb{C}}^*$ for $s\in\mathbb{C}$. The L–functions and ε–factors are the ones defined by Shahidi in Section 7 of [32] and r_i are the irreducible components, indexed as in [32], of the adjoint representation r of the Langlands dual group of the Levi factor M on the Langlands dual Lie algebra of the Lie algebra of the unipotent radical of P. The normalized intertwining operator $N(s\widetilde{\alpha}, \pi_v, w)$ is given by

$$A(s\widetilde{\alpha}, \pi_v, w) = r(s\widetilde{\alpha}, \pi_v, w)N(s\widetilde{\alpha}, \pi_v, w).$$

Once the normalization is defined for the maximal proper parabolic subgroup case, the normalization factor

$$r(\underline{s}, \pi_v, w)$$

for a ψ_v–generic representation π_v of a general proper parabolic subgroup P_θ with the Levi factor M_θ and $\underline{s} \in \mathfrak{a}^*_{M,\mathbb{C}}$ is defined as the product of the normalizing factors for the maximal proper parabolic cases appearing in the decomposition, according to a reduced decomposition of w into simple reflections, of the standard intertwining operator $A(\underline{s}, \pi_v, w)$ given in Section 2.1 of [31]. Although a reduced decomposition of w into simple reflections is not unique, the normalizing factor is independent of the choice of such decomposition. The normalized intertwining operator $N(\underline{s}, \pi_v, w)$ is again defined by

$$A(\underline{s}, \pi_v, w) = r(\underline{s}, \pi_v, w)N(\underline{s}, \pi_v, w).$$

We recall the main result of [39] (see also Proposition 1.3 of [8]) showing the holomorphy and non–vanishing of the normalized intertwining operator $N(\underline{s}, \pi_v, w)$ in a certain open set slightly bigger than the closure of the positive Weyl chamber for a tempered ψ_v–generic representation π_v.

Proposition 1.1. *Let P_θ be the proper parabolic subgroup of G corresponding to a subset θ of the set of simple roots and w an element of the Weyl group W such that $w(\theta)$ is also a subset of the set of simple roots. Let π_v be an irreducible ψ_v–generic tempered representation of the Levi factor $M_\theta(k_v)$. Then the normalized intertwining operator*

$$N(\underline{s}, \pi_v, w)$$

is holomorphic and non–vanishing for $\underline{s} \in \mathfrak{a}^*_{M,\mathbb{C}}$ such that

$$\langle Re(\underline{s}), \alpha^\vee \rangle > -1/\ell_\alpha \quad \text{for all } \alpha \in \Phi^+_{w,\theta},$$

where ℓ_α is the length of the corresponding adjoint representation r_α in the decomposition of the standard intertwining operator given in Section 2.1 of [31] *and $\Phi^+_{\theta,w}$ is the set of all the positive roots α such that $w\alpha$ is a negative root.*

Finally, we have to a consider non–tempered unitary ψ_v–generic representation. This will be done just for representations $\pi_v \cong \sigma_{1,v} \otimes \ldots \otimes \sigma_{n,v}$ of $M_0(k_v) \cong GL_2(k_v) \times \ldots \times GL_2(k_v)$ in the split $SO_{4n}(k_v)$ or $Sp_{4n}(k_v)$.

Proposition 1.2. *Let P_0 be the standard parabolic subgroup of the split group SO_{4n} or Sp_{4n} with the Levi factor $M_0 \cong GL_2 \times \ldots \times GL_2$. Let $\pi_v \cong \sigma_{1,v} \otimes \ldots \otimes \sigma_{n,v}$ be an irreducible unitary non–tempered generic representation of $M_0(k_v)$. Then, for every $w \in W(M_0)$, the normalized intertwining operator*

$$N(\underline{s}, \pi_v, w)$$

is holomorphic and non–vanishing in the closure of the positive Weyl chamber in $\mathfrak{a}^*_\mathbb{C}$, i.e. for all $\underline{s} = (s_1, \ldots, s_n) \in \mathfrak{a}^*_\mathbb{C}$ such that

$$Re(s_1) \geqslant \ldots \geqslant Re(s_n) \geqslant 0.$$

Proof. If a unitary generic representation $\sigma_{i,v}$ of $GL_2(k_v)$ is not tempered, then it is a complementary series, i.e. the fully induced representation of the form

$$\sigma_{i,v} \cong \mathrm{Ind}_{GL_1(k_v)\times GL_1(k_v)}^{GL_2(k_v)}(\mu_{i,v}|\cdot|^{r_i} \otimes \mu_{i,v}|\cdot|^{-r_i}),$$

where $\mu_{i,v}$ is a unitary character of $GL_1(k_v)$ and $0 < r_i < 1/2$. Since the intertwining operators are compatible with the induction in stages the problem of the holomorphy and non–vanishing is reduced to the tempered case.

More precisely, there is a tempered representation τ_v of one of the Levi factors $L(k_v)$ contained in $M_0(k_v)$ and an element $\underline{s}'$ of the corresponding space $\mathfrak{a}^*_{L,\mathbb{C}}$ such that $I(\underline{s}, \pi_v) \cong I(\underline{s}', \tau_v)$. Therefore, the holomorphy and non–vanishing of $N(\underline{s}, \pi_v, w)$ is equivalent to the holomorphy and non–vanishing of $N(\underline{s}', \tau_v, w)$. If $\underline{s} = (s_1, \ldots, s_n)$, then $\underline{s}'$ is obtained by replacing s_i with $(s_i + r_i, s_i - r_i)$ for all i such that $\sigma_{i,v}$ is a complementary series. Now, it is enough to check that if $Re(s_1) \geqslant \ldots Re(s_n) \geqslant 0$, then the inequalities of Proposition 1.1 are satisfied. That is a straightforward check using the bound $0 < r_i < 1/2$. □

At the end of this Subsection we collect the normalizing factors for the generic split maximal proper parabolic cases needed in the sequel. For the case $GL_2 \times GL_2 \subset GL_4$ the normalizing factor of the standard intertwining operator $A((s_1, s_2), \sigma_{1,v} \otimes \sigma_{2,v}, w)$ acting on the induced representation

$$I((s_1, s_2), \sigma_{1,v} \otimes \sigma_{2,v}) = \mathrm{Ind}_{GL_2(k_v)\times GL_2(k_v)}^{GL_4(k_v)}(\sigma_{1,v}\nu^{s_1} \otimes \sigma_{2,v}\nu^{s_2}) \cong I\left((s_1 - s_2)\widetilde{\alpha}, \sigma_{1,v} \otimes \sigma_{2,v}\right)$$

equals

$$r((s_1, s_2), \sigma_{1,v} \otimes \sigma_{2,v}, w) = \frac{L(s_1 - s_2, \sigma_{1,v} \times \widetilde{\sigma}_{2,v})}{L(1 + s_1 - s_2, \sigma_{1,v} \times \widetilde{\sigma}_{2,v})\varepsilon(s_1 - s_2, \sigma_{1,v} \times \widetilde{\sigma}_{2,v}, \psi_v)}, \tag{2}$$

where the L–functions and ε–factors are the Rankin–Selberg ones of pairs and $\widetilde{\ }$ denotes the contragredient representation. For the case $GL_2 \subset SO_4$ the normalizing factor of the standard intertwining operator $A(s, \sigma_v, w)$ acting on the induced representation

$$I(s, \sigma_v) = \mathrm{Ind}_{GL_2(k_v)}^{SO_4(k_v)}(\sigma_v\nu^s) = I(2s\widetilde{\alpha}, \sigma_v)$$

equals

$$r(s, \sigma_v, w) = \frac{L(2s, \omega_{\sigma_v})}{L(1 + 2s, \omega_{\sigma_v})\varepsilon(2s, \omega_{\sigma_v}, \psi_v)}, \tag{3}$$

where the L–functions and ε–factors are the Hecke ones for the central character ω_{σ_v} of σ_v. For the case $GL_2 \subset Sp_4$ the normalizing factor of the standard intertwining operator $A(s, \sigma_v, w)$ acting on the induced representation

$$I(s, \sigma_v) = \mathrm{Ind}_{GL_2(k_v)}^{Sp_4(k_v)}(\sigma_v\nu^s) = I(s\widetilde{\alpha}, \sigma_v)$$

equals

$$r(s, \sigma_v, w) = \frac{L(s, \sigma_v)}{L(1 + s, \sigma_v)\varepsilon(s, \sigma_v, \psi_v)} \cdot \frac{L(2s, \omega_{\sigma_v})}{L(1 + 2s, \omega_{\sigma_v})\varepsilon(2s, \omega_{\sigma_v}, \psi_v)}, \tag{4}$$

where the L–functions and ε–factors are the principal Jacquet ones for σ_v and the Hecke ones for the central character ω_{σ_v} of σ_v.

1.2. **Non–generic split case.** In this Subsection let v be a place of k where D splits. We define the normalization factor for the standard intertwining operators attached to an one–dimensional unitary representation

$$\pi_v \cong (\chi_{1,v} \circ \det_v) \otimes \ldots \otimes (\chi_{n,v} \circ \det_v)$$

of the Levi factor $M_0(k_v) \cong GL_2(k_v) \times \ldots \times GL_2(k_v)$ of the split group $SO_{4n}(k_v)$ or $Sp_{4n}(k_v)$. The strategy follows the Mœglin and Waldspurger proof of Lemma I.8 in [24] rather closely. See also Section 1.2 of [8].

Representation π_v embeds into the induced representation

$$I_T^M(\underline{s}', \tau_v) = \mathrm{Ind}_{T(k_v)}^{M(k_v)} \left(\chi_{1,v}|\cdot|^{-1/2} \otimes \chi_{1,v}|\cdot|^{1/2} \otimes \ldots \otimes \chi_{n,v}|\cdot|^{-1/2} \otimes \chi_{n,v}|\cdot|^{1/2} \right)$$

as the unique irreducible subrepresentation, where T is the maximal split torus in SO_{4n} or Sp_{4n},

$$\underline{s}' = (-1/2, 1/2, \ldots, -1/2, 1/2) \in \mathfrak{a}_{T,\mathbb{C}}^* \quad \text{and} \quad \tau_v = \chi_{1,v} \otimes \chi_{1,v} \otimes \ldots \otimes \chi_{n,v} \otimes \chi_{n,v}.$$

Hence, for every element $w \in W(M_0)$ of the Weyl group the standard intertwining operator $A(\underline{s}, \pi_v, w)$, where $\underline{s} = (s_1, \ldots, s_n) \in \mathfrak{a}_{\mathbb{C}}^*$, fits into the commutative diagram

$$\begin{array}{ccc} I(\underline{s},\pi_v) & \hookrightarrow & I(\underline{s}+\underline{s}',\tau_v) \\ A(\underline{s},\pi_v,w)\Big\downarrow & & \Big\downarrow A(\underline{s}+\underline{s}',\tau_v) \\ I(w(\underline{s}),w(\pi_v)) & \hookrightarrow & I(w(\underline{s}+\underline{s}'),w(\tau_v)). \end{array}$$

Here $\underline{s}$ is embedded into $\mathfrak{a}^*_{T,\mathbb{C}}$. In other words, $A(\underline{s},\pi_v,w)$ is the restriction of $A(\underline{s}+\underline{s}',\tau_v)$ to $I(\underline{s},\pi_v)$. Hence, the normalizing factor for $A(\underline{s},\pi_v,w)$ is defined to be

$$r(\underline{s},\pi_v,w) = r(\underline{s}+\underline{s}',\tau_v,w) \tag{5}$$

and the normalized operator $N(\underline{s},\pi_v,w)$ is actually the restriction of $N(\underline{s}+\underline{s}',\tau_v,w)$ to $I(\underline{s},\pi_v)$.

Proposition 1.3. *For every $w \in W(M_0)$, the normalized intertwining operator*

$$N\left((s_1,\ldots,s_n),(\chi_{1,v}\circ\det_v)\otimes\ldots\otimes(\chi_{n,v}\circ\det_v),w\right)$$

defined above is holomorphic and non–vanishing in the positive open Weyl chamber

$$Re(s_1) > \ldots > Re(s_n) > 0.$$

Proof. Let w' be the element of the Weyl group corresponding to the permutation

$$w' = (1,2)(3,4)\ldots(2n-1,2n),$$

where $(i_1,\ldots,i_l)$ denotes the cycle mapping $i_1 \to i_2 \to \ldots \to i_l \to i_1$. The Weyl group element corresponding to a permutation p acts on $\mathfrak{a}^*_{T,\mathbb{C}}$ as $(s_1,\ldots,s_{2n}) \to (s_{p^{-1}(1)},\ldots,s_{p^{-1}(2n)})$ and analogously on representations.

By the discussion above, π_v is the unique irreducible subrepresentation of $I_T^M(\underline{s}',\tau_v)$. Hence, it is the image of the $M(k_v)$ normalized intertwining operator $N(w'^{-1}(\underline{s}'),w'^{-1}(\tau_v),w')$. Observe that $w'(\underline{s}) = \underline{s}$. Then, $N(\underline{s},\pi_v,w)$ fits into the following commutative diagram

$$\begin{array}{ccc} I(\underline{s},\pi_v) & \xleftarrow{N(\underline{s}+w'^{-1}(\underline{s}'),w'^{-1}(\tau_v),w')} & I(\underline{s}+w'^{-1}(\underline{s}'),w'^{-1}(\tau_v)) \\ {\scriptstyle N(\underline{s},\pi_v,w)}\Big\downarrow & & \Big\downarrow{\scriptstyle N(\underline{s}+w'^{-1}(\underline{s}'),w'^{-1}(\tau_v),ww')} \\ I(w(\underline{s}),w(\pi_v)) & \hookrightarrow & I(w(\underline{s}+\underline{s}'),w(\tau_v)), \end{array}$$

where the upper horizontal arrow is surjective. Since for $\underline{s}$ in the positive open Weyl chamber

$$\underline{s}+w'^{-1}(\underline{s}') = (s_1+1/2,s_1-1/2,\ldots,s_n+1/2,s_n-1/2) \in \mathfrak{a}^*_{T,\mathbb{C}}$$

satisfies the inequalities of Proposition 1.1 for the Weyl group element ww', the right vertical arrow is holomorphic and non–vanishing. Therefore, the commutativity of the diagram implies that $N(\underline{s},\pi_v,w)$ is also holomorphic and non–vanishing for such $\underline{s}$. □

Finally, we collect the normalizing factors in this case for the maximal proper parabolic subgroup cases. The general proper parabolic subgroup normalizing factors are again just the product of those. For the case $GL_2 \times GL_2 \subset GL_4$ the normalizing factor of the standard intertwining operator $A\left((s_1,s_2),(\chi_{1,v}\circ\det_v)\otimes(\chi_{2,v}\circ\det_v),w\right)$ acting on the induced representation

$$\begin{aligned} I((s_1,s_2),(\chi_{1,v}\circ\det_v)\otimes(\chi_{2,v}\circ\det_v)) &= \mathrm{Ind}_{GL_2(k_v)\times GL_2(k_v)}^{GL_4(k_v)}\left((\chi_{1,v}\circ\det_v)\nu^{s_1}\otimes(\chi_{2,v}\circ\det_v)\nu^{s_2}\right) \\ &\cong I\left((s_1-s_2)\widetilde{\alpha},(\chi_{1,v}\circ\det_v)\otimes(\chi_{2,v}\circ\det_v)\right) \end{aligned}$$

equals

$$r((s_1,s_2),(\chi_{1,v}\circ\det_v)\otimes(\chi_{2,v}\circ\det_v),w)=r_1(s_1-s_2,\chi_{1,v}\chi_{2,v}^{-1}), \tag{6}$$

where, for $s\in\mathbb{C}$ and a unitary character χ_v of $k_v^\times$,

$$r_1(s,\chi_v)=\frac{L(s,\chi_v)L(s-1,\chi_v)}{L(s+2,\chi_v)L(s+1,\chi_v)\varepsilon(s+1,\chi_v,\psi_v)\varepsilon(s,\chi_v,\psi_v)^2\varepsilon(s-1,\chi_v,\psi_v)}.$$

The L–functions and ε–factors are the Hecke ones for a character. For the case $GL_2\subset SO_4$ the normalizing factor of the standard intertwining operator $A(s,\chi_v\circ\det_v,w)$ acting on the induced representation

$$I(s,\chi_v\circ\det_v)=\mathrm{Ind}_{GL_2(k_v)}^{SO_4(k_v)}((\chi_v\circ\det_v)\nu^s)=I(2s\widetilde{\alpha},\chi_v\circ\det_v)$$

equals

$$r(s,\chi_v\circ\det_v,w)=\frac{L(2s,\chi_v^2)}{L(2s+1,\chi_v^2)\varepsilon(2s,\chi_v^2,\psi_v)}, \tag{7}$$

where the L–functions and ε–factors are the Hecke ones for the central character χ_v^2 of $\chi_v\circ\det_v$. For the case $GL_2\subset Sp_4$ the normalizing factor of the standard intertwining operator $A(s,\chi_v\circ\det_v,w)$ acting on the induced representation

$$I(s,\sigma_v)=\mathrm{Ind}_{GL_2(k_v)}^{Sp_4(k_v)}((\chi_v\circ\det_v)\nu^s)=I(s\widetilde{\alpha},\chi_v\circ\det_v)$$

equals

$$\begin{aligned}(8)\quad & r(s,\chi_v\circ\det_v,w)=\\ &=\frac{L(s+1/2,\chi_v)}{L(s+3/2,\chi_v)\varepsilon(s+1/2,\chi_v,\psi_v)}\cdot\frac{L(s-1/2,\chi_v)}{L(s+1/2,\chi_v)\varepsilon(s-1/2,\chi_v,\psi_v)}\cdot\frac{L(2s,\chi_v^2)}{L(1+2s,\chi_v^2)\varepsilon(2s,\chi_v^2,\psi_v)},\end{aligned}$$

where the L–functions and ε–factors are the Hecke ones for χ_v and for the central character χ_v^2 of $\chi_v\circ\det_v$.

1.3. **Non–split case.** In this Subsection let v be a place of k where D does not split. We define the normalization factors for the standard intertwining operators attached to irreducible a unitary representation

$$\pi'_v\cong\sigma'_{1,v}\otimes\ldots\otimes\sigma'_{n,v}$$

of the Levi factor $M'_0(k_v)\cong GL'_1(k_v)\times\ldots\times GL'_1(k_v)$ of the group $G'_n(k_v)$ or $H'_n(k_v)$. Since $GL'_1(k_v)$ has no proper parabolic subgroups, π'_v and all $\sigma'_{i,v}$ are supercuspidal.

For $w\in W'$ and $\underline{s}\in\mathfrak{a}^*_{\mathbb{C}}$, the standard intertwining operator $A(\underline{s},\pi'_v,w)$ is defined as the analytic continuation of the usual integral (see for example [30] or Section 1.3 of [8]). Here we have to stress that the Haar measures used in the definition of the standard intertwining operators for the split and non–split case are chosen compatibly as explained in Section 2 of [30]. That enables the transfer of the Plancherel measure between the split case and its non–split inner form as in [30] which is crucial in the proof of the holomorphy and non–vanishing of the normalized intertwining operators defined below.

In the definition of the normalizing factor for $A(\underline{s},\pi'_v,w)$ we use the local lift of representations from $GL'_1(k_v)$ to $GL_2(k_v)$ defined using the local Jacquet–Langlands correspondence at the beginning of this Section. If

$$\pi_v\cong\sigma_{1,v}\otimes\ldots\otimes\sigma_{n,v}$$

denotes the local lift of π'_v from $M'_0(k_v)$ to $M_0(k_v)$, then the normalizing factor is defined to be

$$r(\underline{s}, \pi'_v, w) = r(\underline{s}, \pi_v, w) \tag{9}$$

and the normalized intertwining operator $N(\underline{s}, \pi'_v, w)$ by

$$A(\underline{s}, \pi'_v, w) = r(\underline{s}, \pi'_v, w) N(\underline{s}, \pi'_v, w).$$

Observe that π_v is square–integrable as the local lift of a supercuspidal representation.

Proposition 1.4. *Let $\pi'_v \cong \sigma'_{1,v} \otimes \ldots \otimes \sigma'_{n,v}$ be an irreducible unitary representation of the Levi factor $M'_0(k_v) \cong GL'_1(k_v) \times \ldots \times GL'_1(k_v)$ of $G'_n(k_v)$ or $H'_n(k_v)$ and $\underline{s} = (s_1, \ldots, s_n) \in \mathfrak{a}^*_{\mathbb{C}}$. Then, for every $w \in W'$, the normalized intertwining operator*

$$N(\underline{s}, \pi'_v, w)$$

is holomorphic and non–vanishing in the closure of the positive Weyl chamber

$$Re(s_1) \geqslant \ldots \geqslant Re(s_n) \geqslant 0.$$

Proof. The proof is essentially the same as the proof of Proposition 1.11 in [8]. Decomposition of the standard intertwining operator reduces the proof to the maximal proper parabolic cases treated in Proposition 1.10 of [8] except the new case $GL'_1 \subset H'_1$. But that case is settled in the same way since the result on the Plancherel measure from [30] holds. □

Again, we collect here the normalizing factors for the maximal proper parabolic cases. But in the non–split case the normalizing factors are given using the normalizing factors of the split case. Hence, the maximal parabolic cases are given by equations (2) for $GL'_1 \times GL'_1 \subset GL'_2$, (3) for $GL'_1 \subset G'_1$ and (4) for $GL'_1 \subset H'_1$. Nevertheless, we rewrite these equations in more appropriate manner just for the one–dimensional unitary representations of $GL'_1(k_v)$.

As already mentioned, by Theorem (8.1) of [7], the local lift of an one–dimensional unitary representation $\chi_v \circ \det'_v$ of $GL'_1(k_v)$, where χ_v is a unitary character of $k_v^\times$, is the Steinberg representation St_{χ_v}. Hence, in the equations for the normalizing factor the Rankin–Selberg of pairs and principal Jacquet L–functions and ε–factors for the Steinberg representations appear, as well as the Hecke L–function and ε–factor of the central character of St_{χ_v}. By Theorem (3.1), Sections 8 and 9 of [13] and Section (3.1) of [11] those L–functions and ε–factors can be written as

$$\begin{aligned}
L(s, St_{\chi_{1,v}} \times St_{\chi_{2,v}^{-1}}) &= L(s+1, \chi_{1,v}\chi_{2,v}^{-1}) L(s, \chi_{1,v}\chi_{2,v}^{-1}),\\
\varepsilon(s, St_{\chi_{1,v}} \times St_{\chi_{2,v}^{-1}}, \psi_v) &= \varepsilon(s+1, \chi_{1,v}\chi_{2,v}^{-1}, \psi_v)\varepsilon(s, \chi_{1,v}\chi_{2,v}^{-1}, \psi_v)^2 \varepsilon(s-1, \chi_{1,v}\chi_{2,v}^{-1}, \psi_v) \cdot \\
&\quad \cdot \frac{L(1-s, \chi_{1,v}^{-1}\chi_{2,v}) L(-s, \chi_{1,v}^{-1}\chi_{2,v})}{L(s-1, \chi_{1,v}\chi_{2,v}^{-1}) L(s, \chi_{1,v}\chi_{2,v}^{-1})},\\
L(s, St_{\chi_v}) &= L(s+1/2, \chi_v),\\
\varepsilon(s, St_{\chi_v}, \psi_v) &= \varepsilon(s+1/2, \chi_v, \psi_v)\varepsilon(s-1/2, \chi_v, \psi_v)\frac{L(1/2-s, \chi_v^{-1})}{L(s-1/2, \chi_v)},\\
L(s, \omega_{St_{\chi_v}}) &= L(s, \chi_v^2),\\
\varepsilon(s, \omega_{St_{\chi_v}}, \psi_v) &= \varepsilon(s, \chi_v^2, \psi_v).
\end{aligned}$$

Therefore, for the case $GL'_1 \times GL'_1 \subset GL'_2$ the normalizing factor of the standard intertwining operator $A((s_1, s_2), (\chi_{1,v} \circ \det'_v) \otimes (\chi_{2,v} \circ \det'_v), w)$ acting on the induced representation

$$I((s_1, s_2), (\chi_{1,v} \circ \det'_v) \otimes (\chi_{2,v} \circ \det'_v)) = \mathrm{Ind}_{GL'_1(k_v) \times GL'_1(k_v)}^{GL'_2(k_v)} \left((\chi_{1,v} \circ \det'_v)\nu^{s_1} \otimes (\chi_{2,v} \circ \det'_v)\nu^{s_2}\right)$$

equals

$$r((s_1, s_2), (\chi_{1,v} \circ \det'_v) \otimes (\chi_{2,v} \circ \det'_v), w) = r_2(s_1 - s_2, \chi_{1,v}\chi_{2,v}^{-1}), \tag{10}$$

where, for $s \in \mathbb{C}$ and a unitary character χ_v of $k_v^\times$,

$$r_2(s, \chi_v) = \\ = \frac{L(s+1, \chi_v)L(s, \chi_v)}{L(s+2, \chi_v)L(s+1, \chi_v)\varepsilon(s+1, \chi_v, \psi_v)\varepsilon(s, \chi_v, \psi_v)^2\varepsilon(s-1, \chi_v, \psi_v)} \frac{L(s, \chi_v)L(s-1, \chi_v)}{L(-s, \chi_v^{-1})L(1-s, \chi_v^{-1})}.$$

For the case $GL'_1 \subset G'_1$ the normalizing factor of the standard intertwining operator $A(s, \chi_v \circ \det'_v, w)$ acting on the induced representation

$$I(s, \pi'_v) = \mathrm{Ind}_{GL'_1(k_v)}^{G'_1(k_v)}(\pi'_v\nu^s)$$

equals

$$r(s, \chi_v \circ \det'_v, w) = \frac{L(2s, \chi_v^2)}{L(1+2s, \chi_v^2)\varepsilon(2s, \chi_v^2, \psi_v)}. \tag{11}$$

For the case $GL'_1 \subset H'_1$ the normalizing factor of the standard intertwining operator $A(s, \chi_v \circ \det'_v, w)$ acting on the induced representation

$$I(s, \pi'_v) = \mathrm{Ind}_{GL'_1(k_v)}^{H'_1(k_v)}(\pi'_v\nu^s)$$

equals

$$r(s, \chi_v \circ \det'_v, w) = \\ = \frac{L(s+1/2, \chi_v)}{L(s+3/2, \chi_v)\varepsilon(s+1/2, \chi_v, \psi_v)\varepsilon(s-1/2, \chi_v, \psi_v)} \frac{L(s-1/2, \chi_v)}{L(1/2-s, \chi_v^{-1})} \cdot \frac{L(2s, \chi_v^2)}{L(1+2s, \chi_v^2)\varepsilon(2s, \chi_v^2, \psi_v)}. \tag{12}$$

1.4. **Global normalization.** In this Subsection we combine the local results of the previous Subsections in order to get the normalization factor for the global intertwining operators attached to a cuspidal automorphic representation

$$\pi' \cong \sigma'_1 \otimes \ldots \otimes \sigma'_n$$

of the Levi factor $M'_0(\mathbb{A}) \cong GL'_1(\mathrm{A}) \times \ldots \times GL'_1(\mathbb{A})$ of the group $G'_n(\mathbb{A})$ or $H'_n(\mathbb{A})$, where σ_i are cuspidal automorphic representations of $GL'_1(\mathbb{A})$ which are either all not one–dimensional or all one–dimensional.

For $w \in W'$ and $\underline{s} \in \mathfrak{a}_{\mathbb{C}}^*$, the global standard intertwining operator $A(\underline{s}, \pi', w)$ decomposes according to the restricted tensor product $\pi' \cong \otimes_v \pi'_v$, when acting on a pure tensor $f_{\underline{s}} = \otimes_v f_{\underline{s},v} \in I(\underline{s}, \pi')$, into the product of the local standard intertwining operators $A(\underline{s}, \pi'_v, w)$. Hence, it is natural to define the global normalizing factor to be the product over all places of the local ones, i.e.

$$r(\underline{s}, \pi', w) = \prod_v r(\underline{s}, \pi'_v, w). \tag{13}$$

The holomorphy and non–vanishing of the global normalized intertwining operators $N(\underline{s},\pi',w)$ defined by

$$A(\underline{s},\pi',w)=r(\underline{s},\pi',w)N(\underline{s},\pi',w)$$

'deep enough' in the positive Weyl chamber is proved in the following Theorem.

Theorem 1.5. *Let $\pi'=\sigma_1'\otimes\ldots\otimes\sigma_n'$ be a cuspidal automorphic representation of the Levi factor $M_0'(\mathbb{A})\cong GL_1'(\mathbb{A})\times\ldots\times GL_1'(\mathbb{A})$ of the group $G_n'(\mathbb{A})$ or $H_n'(\mathbb{A})$ such that σ_i' are either all not one–dimensional or all one–dimensional. Then, for every $w\in W'$, the global normalizing factor*

$$r(\underline{s},\pi',w)$$

is a meromorphic function of $\underline{s}$. If all σ_i are not one–dimensional then the global normalized operator

$$N(\underline{s},\pi',w)$$

is holomorphic and non–vanishing for $\underline{s}$ in the closure of the positive Weyl chamber

$$Re(s_1)\geqslant\ldots\geqslant Re(s_n)\geqslant 0.$$

If all σ_i are one–dimensional then the global normalized operator

$$N(\underline{s},\pi',w)$$

is holomorphic and non–vanishing for $\underline{s}$ in the positive open Weyl chamber

$$Re(s_1)>\ldots>Re(s_n)>0.$$

Proof. The local normalizing factors $r(\underline{s},\pi_v',w)$ are defined in terms of the local L–functions and ε–factors. At all split places $v\notin S$ these are the L–functions and ε–factors of the same global cuspidal automorphic representation of a split classical group. Therefore, the product over all split places of the normalizing factors converges absolutely for $\underline{s}$ deep enough in the positive Weyl chamber and its analytic continuation is given using the partial L–functions and ε–factors which are meromorphic. Since the normalizing factors at the remaining finite number of non–split places $v\in S$ are meromorphic, the global normalizing factor $r(\underline{s},\pi',w)$ is meromorphic.

The global normalized intertwining operator decomposes into the tensor product of the local ones. At almost all places the representation π_v' is spherical, and therefore, the local normalized intertwining operator $N(\underline{s},\pi_v',w)$ just sends the suitably normalized invariant vector of $I(\underline{s},\pi_v')$ for the fixed maximal compact subgroup to the suitably normalized invariant one of $I(w(\underline{s}),w(\pi_v'))$. At the remaining finite number of places the local normalized intertwining operator is holomorphic and non–vanishing in the required region by Propositions 1.1, 1.2, 1.3 and 1.4 of the preceding Subsections. Hence, the global normalized intertwining operator $N(\underline{s},\pi',w)$ is holomorphic and non–vanishing in the required region. □

At the end of every Subsection above we collected the local normalizing factors for the maximal proper parabolic subgroup cases. Here we collect the global normalizing factors for those cases. First, assume that all σ_i' are not one–dimensional representations of $GL_1'(\mathbb{A})$. Then, by our definition at the beginning of this Section, the local lift $\sigma_{i,v}$ of $\sigma_{i,v}'$ from $GL_1'(k_v)$ to $GL_2(k_v)$ is compatible with the global lift σ_i. Moreover, σ_i is a cuspidal automorphic representation of $GL_2(\mathbb{A})$. Therefore, the product over all places of the local normalizing factors in equations (2), (3), (4) and (9) can be written using the global L–functions and ε–factors attached to cuspidal automorphic representations. For the case $GL_1' \times GL_1' \subset GL_2'$ the global normalizing factor of the standard global intertwining operator $A((s_1, s_2), \sigma_1' \otimes \sigma_2', w)$ acting on the induced representation

$$I((s_1, s_2), \sigma_1' \otimes \sigma_2') = \mathrm{Ind}_{GL_1'(\mathbb{A}) \times GL_1'(\mathbb{A})}^{GL_2'(\mathbb{A})} \left(\sigma_1' \nu^{s_1} \otimes \sigma_2' \nu^{s_2}\right)$$

equals

$$r((s_1, s_2), \sigma_1' \otimes \sigma_2', w) = \frac{L(s_1 - s_2, \sigma_1 \times \widetilde{\sigma}_2)}{L(1 + s_1 - s_2, \sigma_1 \times \widetilde{\sigma}_2)\varepsilon(s_1 - s_2, \sigma_1 \times \widetilde{\sigma}_2)} \tag{14}$$

where the L–functions and ε–factors are the global Rankin–Selberg ones of pairs. For the case $GL_1' \subset G_1'$ the global normalizing factor of the standard global intertwining operator $A(s, \sigma', w)$ acting on the induced representation

$$I(s, \sigma') = \mathrm{Ind}_{GL_1'(\mathbb{A})}^{G_1'(\mathbb{A})} \left(\sigma' \nu^s\right)$$

equals

$$r(s, \sigma', w) = \frac{L(2s, \omega_\sigma)}{L(1 + 2s, \omega_\sigma)\varepsilon(2s, \omega_\sigma)} \tag{15}$$

where the L–functions and ε–factors are the global Hecke ones for the central character ω_σ of σ. For the case $GL_1' \subset H_1'$ the global normalizing factor of the standard global intertwining operator $A(s, \sigma', w)$ acting on the induced representation

$$I(s, \sigma') = \mathrm{Ind}_{GL_1'(\mathbb{A})}^{H_1'(\mathbb{A})} \left(\sigma' \nu^s\right)$$

equals

$$r(s, \sigma', w) = \frac{L(s, \sigma)}{L(1 + s, \sigma)\varepsilon(s, \sigma)} \cdot \frac{L(2s, \omega_\sigma)}{L(1 + 2s, \omega_\sigma)\varepsilon(2s, \omega_\sigma)} \tag{16}$$

where the L–functions and ε–factors are the global principal Jacquet ones for σ and the Hecke ones for the central character ω_σ of σ.

Next, assume that all σ_i' are one–dimensional cuspidal automorphic representations of $GL_1'(\mathbb{A})$, i.e. $\sigma_i' = \chi_i \circ \det'$, where χ_i are unitary characters of $\mathbb{A}^\times / k^\times$. Now, the local and global lift are not compatible, and hence, the local normalizing factors at split places in equations (6), (7) and (8) are not of the same form as the local normalizing factors at non–split places in equations (10), (11) and (12). Therefore, in the global normalization factors for maximal proper parabolic cases, along with global Hecke L–functions and ε–factors, the local Hecke L–functions appear.

For the case $GL_1' \times GL_1' \subset GL_2'$ the global normalizing factor of the standard global intertwining operator $A((s_1, s_2), (\chi_1 \circ \det') \otimes (\chi_2 \circ \det'), w)$ acting on the induced representation

$$I((s_1, s_2), (\chi_1 \circ \det') \otimes (\chi_2 \circ \det')) = \mathrm{Ind}_{GL_1'(\mathbb{A}) \times GL_1'(\mathbb{A})}^{GL_2'(\mathbb{A})} \left((\chi_1 \circ \det')\nu^{s_1} \otimes (\chi_2 \circ \det')\nu^{s_2}\right)$$

equals

$$r((s_1, s_2), (\chi_1 \circ \det') \otimes (\chi_2 \circ \det'), w) = r(s_1 - s_2, \chi_1 \chi_2^{-1}), \tag{17}$$

where, for $s \in \mathbb{C}$ and a unitary character χ of $\mathbb{A}^\times / k^\times$,

$$r(s, \chi) = \frac{L(s, \chi)L(s - 1, \chi)}{L(s + 2, \chi)L(s + 1, \chi)\varepsilon(s + 1, \chi)\varepsilon(s, \chi)^2\varepsilon(s - 1, \chi)} \prod_{v \in S} \frac{L(s + 1, \chi_v)L(s, \chi_v)}{L(1 - s, \chi_v^{-1})L(-s, \chi_v^{-1})},$$

and the L–functions and ε–factors are the global and local Hecke ones. For the case $GL_1' \subset G_1'$ the global normalizing factor of the standard global intertwining operator $A(s, \chi \circ \det', w)$ acting on the induced representation

$$I(s, \chi \circ \det') = \mathrm{Ind}_{GL_1'(\mathbb{A})}^{G_1'(\mathbb{A})} \left((\chi \circ \det') \nu^s \right)$$

equals

$$r(s, \chi \circ \det', w) = \frac{L(2s, \chi^2)}{L(1 + 2s, \chi^2) \varepsilon(2s, \chi^2)} \tag{18}$$

where the L–functions and ε–factors are the global Hecke ones. For the case $GL_1' \subset H_1'$ the global normalizing factor of the standard global intertwining operator $A(s, \chi \circ \det', w)$ acting on the induced representation

$$I(s, \chi \circ \det') = \mathrm{Ind}_{GL_1'(\mathbb{A})}^{H_1'(\mathbb{A})} \left((\chi \circ \det') \nu^s \right)$$

equals
(19)

$$r(s, \chi \circ \det', w) = \frac{L(2s, \chi^2)}{L(1 + 2s, \chi^2) \varepsilon(2s, \chi^2)} \frac{L(s - 1/2, \chi)}{L(s + 3/2, \chi) \varepsilon(s + 1/2, \chi) \varepsilon(s - 1/2, \chi)} \prod_{v \in S} \frac{L(s + 1/2, \chi_v)}{L(1/2 - s, \chi_v^{-1})}$$

where the L–functions and ε–factors are the global and local Hecke ones.

2. Construction

In this Section we construct the certain parts of the residual spectrum of the hermitian quaternionic classical groups $G_n'(\mathbb{A})$ and $H_n'(\mathbb{A})$, as well as the split groups $SO_{4n}(\mathbb{A})$ and $Sp_{4n}(\mathbb{A})$. The residual spectrum of a reductive algebraic group is decomposed using the Langlands spectral theory developed in [20]. See also [25]. Briefly stated, the decomposition of the part of the residual spectrum of a reductive algebraic group supported in a proper parabolic subgroup is realized as the sum of the spaces of automorphic forms obtained after the iterated cancellation of the poles inside the closure of the positive Weyl chamber of the Eisenstein series attached to cuspidal automorphic representations of the Levi factor. The constant term map shows that the analytic properties of the Eisenstein series such as the position and order of the poles coincide with the properties of the constant term of the Eisenstein series. On the other hand, the constant term equals the sum of the standard intertwining operators

$$\sum_{w \in W(M)} A(\underline{s}, \pi, w), \tag{20}$$

where M is the Levi factor of a selfconjugate parabolic subgroup, $W(M)$ the normalizer of M modulo M, π a cuspidal automorphic representation of $M(\mathbb{A})$ and $\underline{s} \in \mathfrak{a}_{M,\mathbb{C}}^*$. The assumption that a parabolic subgroup is selfconjugate simplifies the notation and makes no harm since in our case M_0' is the Levi factor of a selfconjugate parabolic subgroup of G_n' or H_n'. In order to study the poles inside the positive Weyl chamber of the sum (20) we use the normalization of the standard intertwining operators of Section 1.

Before passing to the calculation of the residual spectrum in Theorem 2.2 below, we collect the well known analytic properties of the global and local L–functions involved. The proof for the Hecke L–functions can be found in [37], for the principal Jacquet L–functions for GL_2 in [12] and for the Rankin–Selberg L–functions of pairs for $GL_2 \times GL_2$ in [10]. Observe that the global Hecke L–function $L(s, \mathbf{1})$ for the trivial character $\mathbf{1}$ of $\mathbb{A}^\times/k^\times$ is nothing else than the complete ζ–function of the algebraic number field k.

Lemma 2.1. *The global Rankin–Selberg L–function $L(s, \pi_1 \times \pi_2)$ of cuspidal automorphic representations π_1 and π_2 of $GL_2(\mathbb{A})$ has the simple poles at $s = 0$ and $s = 1$ if $\pi_1 \cong \widetilde{\pi}_2$ and it is entire otherwise. It has no zeroes for $Re(s) \geqslant 1$.*

The global principal Jacquet L–function $L(s, \pi)$ of a cuspidal automorphic representation π of $GL_2(\mathbb{A})$ is entire. It has no zeroes for $Re(s) \geqslant 1$.

The global Hecke L–function $L(s, \mu)$ of a unitary character μ of $\mathbb{A}^\times/k^\times$ has the simple poles at $s = 0$ and $s = 1$ if μ is trivial and it is entire otherwise. It has no zeroes for $Re(s) \geqslant 1$. The local Hecke L–function $L(s, \mu_v)$ of a unitary character μ_v of $k_v^\times$ has the real simple pole at $s = 0$ if μ_v is trivial and it is entire otherwise. It has no zeroes.

Theorem 2.2. *Let*

$$\pi' \cong \sigma_1' \otimes \ldots \otimes \sigma_n'$$

be a cuspidal automorphic representation of the Levi factor $M_0'(\mathbb{A}) \cong GL_1'(\mathbb{A}) \times \ldots \times GL_1'(\mathbb{A})$ of the minimal parabolic subgroup of the group $G_n'(\mathbb{A})$ or $H_n'(\mathbb{A})$ such that one of the following holds:

(i) *all σ_i' are not one–dimensional cuspidal automorphic representations of $GL_1'(\mathbb{A})$ with the unitary central character,*
(ii) *all $\sigma_i' \cong \chi_i \circ \det'$ are one–dimensional cuspidal automorphic representations of $GL_1'(\mathbb{A})$, where χ_i is a unitary character of $\mathbb{A}^\times/k^\times$ and for the group H_n' the character χ_n is nontrivial,*
(iii) *the group is H_n', all $\sigma_i' \cong \chi_i \circ \det'$ are one–dimensional cuspidal automorphic representations of $GL_1'(\mathbb{A})$, where χ_i is a unitary character of $\mathbb{A}^\times/k^\times$ and χ_n is trivial.*

Let $\underline{s}_0 \in \mathfrak{a}_{\mathbb{C}}^*$ be

$$\underline{s}_0 = \begin{cases} (n-1/2, \ldots, 3/2, 1/2), & \text{in case (i)}, \\ (2n-3/2, \ldots, 5/2, 1/2), & \text{in case (ii)}, \\ (2n-1/2, \ldots, 7/2, 3/2), & \text{in case (iii)}, \end{cases}$$

and we consider only the part of the residual spectrum obtained as the iterated residue of the Eisenstein series at $\underline{s}_0$. The Eisenstein series attached to π' has the simple iterated pole at $\underline{s}_0$ if and only if

$$\begin{cases} \text{in case (i) for the group } G_n' \text{ all } \sigma_i' \text{ are isomorphic and have the trivial central character,} \\ \text{in case (i) for the group } H_n' \text{ all } \sigma_i' \text{ are isomorphic, have the trivial central character and} \\ \qquad L(1/2, \sigma_i) \neq 0, \text{ where } \sigma_i \text{ is the global lift of } \sigma_i', \\ \text{in case (ii) for the group } G_n' \text{ all } \chi_i \text{ are equal and } \chi_i^2 \text{ is trivial,} \\ \text{in case (ii) for the group } H_n' \text{ all } \chi_i \text{ are equal, } \chi_i^2 \text{ is trivial and} \\ \qquad \chi_{i,v} \text{ is nontrivial at every place } v \in S, \\ \text{in case (iii) all } \chi_i \text{ are trivial.} \end{cases}$$

The space of automorphic forms spanned by the iterated residue at $\underline{s}_0$ is a constituent of the residual spectrum which is denoted by $\mathcal{A}(\pi')$. In all cases, by the constant term map, $\mathcal{A}(\pi')$ is isomorphic to the image of the normalized intertwining operator

$$N(\underline{s}_0, \pi', w_l),$$

where $w_l \in W'$ is the longest Weyl group element. The image is irreducible except for the group H'_n in case (ii) *when the image is the sum of the irreducible representations of the form*

$$\otimes_v \Pi'_v,$$

where Π'_v is the irreducible image of $N(\underline{s}_0, \pi'_v, w_l)$ for $v \in S$ and Π'_v is one of at most two non–isomorphic irreducible components of the image of $N(\underline{s}_0, \pi'_v, w_l)$ for $v \notin S$ and it is unramified for almost all v.

Proof. The poles of the Eisenstein series coincide with the poles of its constant term, i.e. the sum (20) of the standard intertwining operators

$$\sum_{w \in W'} A(\underline{s}, \pi', w), \tag{21}$$

where $\underline{s} \in \mathfrak{a}^*_{\mathbb{C}}$. In Section 1 we normalized the standard intertwining operators using the scalar meromorhic normalizing factors $r(\underline{s}, \pi', w)$. The main result is Theorem 1.5 showing that the normalized intertwining operators are holomorphic and non–vanishing in the positive open Weyl chamber of $\mathfrak{a}^*_{\mathbb{C}}$, i.e. for all $\underline{s} = (s_1, \ldots, s_n) \in \mathfrak{a}^*_{\mathbb{C}}$ such that

$$Re(s_1) > Re(s_2) > \ldots > Re(s_n) > 0.$$

Observe that $\underline{s}_0$ is in the positive open Weyl chamber and hence the poles of the terms in (21) are the poles of their normalizing factors.

The normalizing factor $r(\underline{s}, \pi', w)$ is given as a product of the normalizing factors for the maximal proper parabolic cases appearing in the decomposition of the standard intertwining operator $A(\underline{s}, \pi', w)$ as in Section 2.1 of [31] according to a reduced decomposition of the Weyl group element w into simple reflections. Although the reduced decomposition of the Weyl group element is not unique, the obtained normalizing factor is independent of the chosen reduced decomposition. Therefore, let us fix an algorithm for decomposing the elements of the Weyl group W' by specifying its action on $\mathfrak{a}^*_{\mathbb{C}}$ and on representations of $M'_0(\mathbb{A})$. It is well–known that $W' \cong S_n \ltimes {C_2}^n$, where S_n is the group of permutations of n letters and C_2 the multiplicative group $\{\pm 1\}$. The action of the Weyl group element $w = (p, c)$, where $p \in S_n$ and $c = (c_1, \ldots, c_n) \in {C_2}^n$, on $\underline{s} = (s_1, \ldots, s_n) \in \mathfrak{a}^*_{\mathbb{C}}$ is given by

$$w(\underline{s}) = \left(c_{p^{-1}(1)} s_{p^{-1}(1)}, \ldots, c_{p^{-1}(n)} s_{p^{-1}(n)}\right),$$

and on a representation $\pi' \cong \sigma'_1 \otimes \ldots \otimes \sigma'_n$ of $M'_0(\mathbb{A})$ by

$$w(\pi') = {\sigma'}^{c_{p^{-1}(1)}}_{p^{-1}(1)} \otimes \ldots \otimes {\sigma'}^{c_{p^{-1}(n)}}_{p^{-1}(n)},$$

where ${\sigma'}^1_i = \sigma'_i$ and ${\sigma'}^{-1}_i = \widetilde{\sigma}'_i$. Let

$$I^+_w = \{j \in \{1, \ldots, n\} : c_j = 1\} \quad \text{and} \quad I^-_w = \{j \in \{1, \ldots, n\} : c_j = -1\}.$$

The simple reflections in the Weyl group W' correspond to the transpositions $w_j = (j, j+1) \in S_n$ for $j = 1, \ldots, n-1$ and the element $w_0 = (1, \ldots, 1, -1) \in {C_2}^n$. The algorithm for a reduced decomposition of $w = (p, c) \in W'$ is as follows:

(1) Using the transpositions w_j move the representation with the maximal index in I^-_w to the most right position. In this step transpositions interchange the position of a representation with the index in I^-_w and either a representation with the higher index in I^+_w or the contragredient of a representation with the higher index in I^-_w

(2) Apply w_0 to take the contragredient of the representation moved to the most right position in step (1).
(3) Repeat steps (1) and (2) with all the other elements of I_w^- always choosing the maximal element not used in the previous steps. As a result we obtain the representations with indices in I_w^+ on the left with increasing indices and the contragredients of the representations with indices in I_w^- on the right with decreasing indices.
(4) Using the minimal number of the transpositions w_j arrange the representations with indices in I_w^+ to be ordered as required by the action of w but still all of them on the left, i.e. keeping the contragredients of the representations with indices in I_w^- fixed. In this step every transposition interchanges the position of a representation with index in I_w^+ and a representation with the higher index in I_w^+
(5) As in step (4), using the minimal number of the transpositions w_j arrange the contragredients of the representations with indices in I_w^- to be ordered as required by the action of w but still all of them on the right, i.e. keeping the representations with indices in I_w^+ fixed. In this step every transposition interchanges the position of the contragredient of a representation with index in I_w^- and the contragredient of a representation with the lower index in I_w^-
(6) Using the minimal number of the transpositions w_j arrange all the indices as required by the action of w. In this step every transposition interchanges the position of a representation with the index in I_w^+ and the contragredient of a representation with the index in I_w^-

Now, the normalizing factors appearing in the maximal proper parabolic cases corresponding to the simple reflections in the above steps are given at the end of Subsection 1.4 in equations (14), (15) and (16) for case (i) and in equations (17), (18) and (19) for cases (ii) and (iii). Using the analytic properties of the L–functions involved given in Lemma 2.1, we see that the possible singular hyperplanes of the normalization factors in case (i) are

$$\begin{aligned} s_i - s_j &= 1, \quad \text{for } 1 \leqslant i < j \leqslant n, \\ s_i + s_j &= 1, \quad \text{for } 1 \leqslant i < j \leqslant n, \\ 2s_i &= 1, \quad \text{for } 1 \leqslant i \leqslant n, \end{aligned}$$

and in cases (ii) and (iii)

$$\begin{aligned} s_i - s_j &= 2, \quad \text{for } 1 \leqslant i < j \leqslant n, \\ s_i + s_j &= 2, \quad \text{for } 1 \leqslant i < j \leqslant n, \\ 2s_i &= b_i, \quad \text{for } 1 \leqslant i \leqslant n, \end{aligned}$$

where $b_i = 3$ if the group is H'_n and χ_i is trivial, while $b_i = 1$ otherwise. At all the possible singular hyperplanes the pole is at most simple. Observe that $\underline{s}_0$ in all cases is the intersection of precisely n among the possible singular hyperplanes. Moreover, it is the so called regular point since there are no possible poles of the Eisenstein series deeper in the positive Weyl chamber. The hyperplanes intersecting at $\underline{s}_0$ in case (i) are

$$\begin{aligned} s_i - s_{i+1} &= 1, \quad \text{for } i = 1, \dots, n-1 \\ 2s_n &= 1, \end{aligned}$$

in case (ii)

$$\begin{aligned} s_i - s_{i+1} &= 2, \quad \text{for } i = 1, \ldots, n-1 \\ 2s_n &= 1, \end{aligned}$$

and in case (iii)

$$\begin{aligned} s_i - s_{i+1} &= 2, \quad \text{for } i = 1, \ldots, n-1 \\ 2s_n &= 3. \end{aligned}$$

In order to have the iterated pole of the Eisenstein series at $\underline{s}_0$, all n possible singular hyperplanes intersecting at $\underline{s}_0$ have to be singular. The hyperplane of the form $s_i - s_{i+1} = a$, where $a \in \{1, 2\}$, is singular for the intertwining operator $A(\underline{s}, \pi', w)$ if and only if $\sigma'_i \cong \sigma'_{i+1}$ and in the reduced decomposition of w the interchange of the positions of either σ'_i and σ'_{i+1}, or $\widetilde{\sigma}'_{i+1}$ and $\widetilde{\sigma}'_i$, occurs. The hyperplane $2s_n = 1$ in case (i) for the group G'_n is singular for the intertwining operator $A(\underline{s}, \pi', w)$ if and only if σ'_n has the trivial central character and $c_n = -1$, where $w = (p, c)$. The hyperplane $2s_n = 1$ in case (i) for the group H'_n is singular for the intertwining operator $A(\underline{s}, \pi', w)$ if and only if σ'_n has the trivial central character, the global L–function $L(1/2, \sigma_n) \neq 0$ for the global lift σ_n of σ'_n and $c_n = -1$, where $w = (p, c)$. The hyperplane $2s_n = 1$ in case (ii) for the group G'_n is singular for the intertwining operator $A(\underline{s}, \pi', w)$ if and only if χ_n^2 is trivial and $c_n = -1$, where $w = (p, c)$. The hyperplane $2s_n = 1$ in case (ii) for the group H'_n is singular for the intertwining operator $A(\underline{s}, \pi', w)$ if and only if χ_n^2 is trivial, $\chi_{n,v}$ is nontrivial for all places $v \in S$ and $c_n = -1$, where $w = (p, c)$. The hyperplane $2s_n = 3$ in case (iii) is singular for the intertwining operator $A(\underline{s}, \pi', w)$ if and only if χ_n is trivial and $c_n = -1$, where $w = (p, c)$. Therefore, the necessary conditions for the pole of the Eisenstein series at $\underline{s}_0$ are as claimed in the Theorem.

Assume that the necessary condition for the pole holds. Looking at the reduced decomposition algorithm we see that, in order to get the singular hyperplane of the form $s_i - s_{i+1} = a$ for the intertwining operator corresponding to $w \in W'$, both i and $i+1$ have to be elements of the same set I_w^+ or I_w^-. Since $c_n = -1$ in order to get the singular hyperplane $2s_n = b$, we conclude $n \in I_w^-$. Therefore, $I_w^- = \{1, \ldots, n\}$ and I_w^+ is empty, i.e. if $w = (p, c)$ then $c = (-1, \ldots, -1)$. For such w, during the first three steps of the reduced decomposition algorithm the singular hyperplanes of the form $s_i - s_{i+1} = a$ do not occur. Afterwards, in step (5) we have to obtain all such hyperplanes and hence p must be the identity permutation id.

Therefore, if the necessary condition for the pole holds, the Eisenstein series indeed has the pole at $\underline{s}_0$ and the only element of the Weyl group W' such that the corresponding intertwining operator in (21) has the iterated pole at $\underline{s}_0$ is the longest element $w_l = (id, (-1, \ldots, -1))$. The iterated residue of the constant term is, up to the nonzero constant, equal to the image of the normalized intertwining operator $N(\underline{s}_0, \pi', w_l)$.

The square integrability of the obtained space of automorphic forms follows from the Langlands square integrability criterion from page 104 of [20] because

$$w_l(\underline{s}_0) = \begin{cases} (-(n-1/2), \ldots, -3/2, -1/2), & \text{in case (i)}, \\ (-(2n-3/2), \ldots, -5/2, -1/2), & \text{in case (ii)}, \\ (-(2n-1/2), \ldots, -7/2, -3/2), & \text{in case (iii)}, \end{cases}$$

and, by the criterion, if $w_l(\underline{s}_0) = (s'_1, \ldots, s'_n)$, then the square integrability condition is

$$\sum_{i=1}^{j} s'_i < 0 \qquad \forall j = 1, \ldots, n.$$

Thus the iterated residue at $\underline{s}_0$ of the Eisenstein series attached to π' gives a constituent $\mathcal{A}(\pi')$ of the residual spectrum.

It remains to describe the image of $N(\underline{s}_0, \pi', w_l)$ which is done for every place v of k separately. If π'_v is tempered then the image of $N(\underline{s}_0, \pi'_v, w_l)$ is irreducible by the Langlands classification since $\underline{s}_0$ is in the open positive Weyl chamber and w_l is the longest element of the Weyl group W'. Observe that this is always the case if $v \in S$.

Let $v \notin S$ and assume π'_v is not tempered. Since all σ'_i are isomorphic, in case (i) this means that $\sigma_{i,v}$ is a complementary series representation of $GL_2(k_v)$, i.e. the fully induced representation of the form

$$\sigma_{i,v} \cong \mathrm{Ind}_{GL_1(k_v)\times GL_1(k_v)}^{GL_2(k_v)} \left(\chi_v|\cdot|^{-r} \otimes \chi_v|\cdot|^{r}\right),$$

where χ_v is a unitary character of $k_v^\times$ and $0 < r < 1/2$. In cases (ii) and (iii), $\sigma_{i,v} \cong \chi_v \circ \det_v$ is an one–dimensional representation of $GL_2(k_v)$, i.e. the unique irreducible subrepresentation of the induced representation

$$\mathrm{Ind}_{GL_1(k_v)\times GL_1(k_v)}^{GL_2(k_v)} \left(\chi_v|\cdot|^{-r} \otimes \chi_v|\cdot|^{r}\right),$$

where $r = 1/2$. If we denote by

$$\tau_v \cong \chi_v \otimes \ldots \otimes \chi_v$$

the representation of the maximal split torus $T(k_v) \cong GL_1(k_v) \times \ldots \times GL_1(k_v)$ of $SO_{4n}(k_v)$ or $Sp_{4n}(k_v)$,

$$\underline{s}'_0 = (r, -r, r, -r, \ldots, r, -r) \in \mathfrak{a}^*_{T,\mathbb{C}} \quad \text{and} \quad w' = (1,2)(3,4)\ldots(n-1,n) \in S_{2n}$$

the element of the absolute Weyl group of $SO_{4n}(k_v)$ or $Sp_{4n}(k_v)$, then the image of

$$N(\underline{s}_0 + \underline{s}'_0, \tau_v, w')$$

is isomorphic to $I(\underline{s}_0, \pi')$. Therefore, the image of $N(\underline{s}_0, \pi'_v, w_l)$ is isomorphic to the image of

$$N(\underline{s}_0 + \underline{s}'_0, \tau_v, w_l w').$$

Now, τ_v is tempered and $w_l w'$ is the longest element of the Weyl group for $SO_{4n}(k_v)$ or $Sp_{4n}(k_v)$. Hence, if $\underline{s}_0 + \underline{s}'_0$ is in the open positive Weyl chamber the image is irreducible by the Langlands classification. This is the case except for the group $Sp_{4n}(k_v)$ in case (ii).

Finally, let the group be $Sp_{4n}(k_v)$ in case (ii). Then

$$\underline{s}_0 + \underline{s}'_0 = (2n-1, 2n-2, \ldots, 1, 0)$$

which is not in the open positive Weyl chamber for Sp_{4n}. Writing the longest Weyl group element $w_l w'$ for the Weyl group of Sp_{4n} as $w_l w' = w_1 w_0$ where w_1 is the longest element of the Weyl group modulo the Levi factor isomorphic to $GL_1 \times \ldots \times GL_1 \times SL_2$ and $w_0 = (1, \ldots, 1, -1)$ the simple reflection corresponding to the root $2e_{2n}$ of Sp_{4n}, the normalized intertwining operator decomposes into

$$N(\underline{s}_0 + \underline{s}'_0, \tau_v, w_l w') = N(\underline{s}_0 + \underline{s}'_0, \tau_v, w_1) N(\underline{s}_0 + \underline{s}'_0, \tau_v, w_0).$$

The operator $N(\underline{s}_0+\underline{s}_0', \tau_v, w_0)$ is actually the $SL_2(k_v)$ intertwining operator acting on the induced representation

$$\mathrm{Ind}_{GL_1(k_v)}^{SL_2(k_v)} \chi_v \cong \tau_v^+ \oplus \tau_v^-,$$

which is the sum of at most two irreducible tempered components $\tau_v^{\pm}$ where the sign in the superscript denotes the sign of the action of $N(\underline{s}_0+\underline{s}_0', \tau_v, w_0)$ and τ_v^- is trivial if and only if χ_v is trivial. Then, the image of $N(\underline{s}_0+\underline{s}_0', \tau_v, w_l w')$ is the sum of the images of $N(\underline{s}_0+\underline{s}_0', \tau_v, w_1)$ acting on the two induced representations

$$\mathrm{Ind}_{GL_1(k_v)\times\ldots\times GL_1(k_v)\times SL_2(k_v)}^{Sp_{4n}(k_v)} \left(\chi_v|\cdot|^{2n-1}\otimes\chi_v|\cdot|^{2n-2}\otimes\ldots\otimes\chi_v|\cdot|^{2}\otimes\chi_v|\cdot|\otimes\tau_v^{\pm}\right)$$

which we denote by $\Pi_v^{\pm}$, and Π_v^- is trivial if and only if χ_v is trivial. These images are irreducible by the Langlands classification since w_1 is the longest Weyl group element modulo $GL_1\times\ldots\times GL_1\times SL_2$, $\chi_v\otimes\ldots\otimes\chi_v\otimes\tau_v^{\pm}$ is tempered and $(2n-1, 2n-2, \ldots, 1)$ is in the open positive Weyl chamber. Observe that Π_v^+ is unramified at unramified places. Therefore, the irreducible representation Π_v' in the statement of the Theorem is one of the representations $\Pi_v^{\pm}$ and it is Π_v^+ for almost all v. □

Theorem 2.2 in fact gives the decomposition of the parts of the residual spectrum of $G_n'(\mathbb{A})$ and $H_n'(\mathbb{A})$ obtained as the iterated residues at $\underline{s}_0$ of the Eisenstein series attached to cuspidal automorphic representations π' of the Levi factor $M_0'(\mathbb{A})$ of the minimal parabolic subgroup such that either all σ_i' are not one–dimensional, or all σ_i' are one–dimensional. Denote those parts of the residual spectrum by L^2 for both $G_n'(\mathbb{A})$ and $H_n'(\mathbb{A})$. It will be clear from the context to which group we refer. Then L^2 decomposes according to the cases in Theorem 2.2 into

$$L^2 \cong \begin{cases} L^2_{(\mathrm{i})}\oplus L^2_{(\mathrm{ii})}, & \text{for the group } G_n', \\ L^2_{(\mathrm{i})}\oplus L^2_{(\mathrm{ii})}\oplus L^2_{(\mathrm{iii})}, & \text{for the group } H_n'. \end{cases}$$

where every component denotes the part of the residual spectrum at $\underline{s}_0$ coming from the cuspidal automorphic representations of the corresponding case. Now, Theorem 2.2 gives the decompositions of the following Corollary.

Corollary 2.3. *In the notation as above,*

$$L^2_{(\mathrm{i})} \cong \oplus_{\pi'}\mathcal{A}(\pi'),$$

where the sum is over all case (i) *cuspidal automorphic representations π' of $M_0'(\mathbb{A})$ such that for the group G_n' all σ_i' are isomorphic and have the trivial central character, while for the group H_n' all σ_i' are isomorphic, have the trivial central character and $L(1/2,\sigma_i)\neq 0$ where σ_i is the global lift of σ_i'.*

$$L^2_{(\mathrm{ii})} \cong \oplus_{\pi'}\mathcal{A}(\pi'),$$

where the sum is over all case (ii) *cuspidal automorphic representations π' of $M_0'(\mathbb{A})$ such that for the group G_n' all χ_i are equal and χ_i^2 is trivial, while for the group H_n' all χ_i are equal, χ_i^2 is trivial and $\chi_{i,v}$ is nontrivial for every $v\in S$.*

$$L^2_{(\mathrm{iii})} \cong \mathcal{A}(\mathbf{1}_{M_0'}),$$

where $\mathbf{1}_{M_0'}\cong(\mathbf{1}\circ\det')\otimes\ldots\otimes(\mathbf{1}\circ\det')$ is the trivial representation of $M_0'(\mathbb{A})$.

Next, we introduce a similar notation for the split groups SO_{4n} and Sp_{4n}. Let $L^2_{M_0}$ be just the part of the residual spectrum of $SO_{4n}(\mathbb{A})$ and $Sp_{4n}(\mathbb{A})$ obtained as the iterated residue at

$$\underline{t}_0 = (n-1/2, \ldots, 3/2, 1/2) \in \mathfrak{a}^*_{M_0,\mathbb{C}}$$

of the Eisenstein series attached to cuspidal automorphic representations of the Levi factor $M_0(\mathbb{A}) \cong GL_2(\mathbb{A}) \times \ldots \times GL_2(\mathbb{A})$. Note that $L^2_{M_0}$ is not the full residual spectrum with the cuspidal support in $M_0(\mathbb{A})$. For the group SO_{4n} let $L^2_T(SO_{4n})$ be the part of the residual spectrum obtained as the iterated residue at

$$\underline{t}_0 = (2n-1, \ldots, 1, 0) \in \mathfrak{a}^*_{T,\mathbb{C}}$$

of the Eisenstein series attached to cuspidal automorphic representations of the maximal split torus $T(\mathbb{A}) \cong GL_1(\mathbb{A}) \times \ldots \times GL_1(\mathbb{A})$. For the group Sp_{4n} let $L^2_T(Sp_{4n})$ denote the part of the residual spectrum obtained as the iterated residue at one of the points

$$\underline{t}_0 = \begin{cases} (2n-1, \ldots, 1, 0) \in \mathfrak{a}^*_{T,\mathbb{C}}, \\ (2n, \ldots, 2, 1) \in \mathfrak{a}^*_{T,\mathbb{C}}, \end{cases}$$

of the Eisenstein series attached to cuspidal automorphic representations of the maximal split torus $T(\mathbb{A}) \cong GL_1(\mathbb{A}) \times \ldots \times GL_1(\mathbb{A})$. Again, note that $L^2_T(SO_{4n})$ and $L^2_T(Sp_{4n})$ are not the full residual spectra supported in the torus.

Theorem 2.4. *In the notation as above, the part $L^2_{M_0}$ of the residual spectrum of $SO_{4n}(\mathbb{A})$ or $Sp_{4n}(\mathbb{A})$ decomposes into*

$$L^2_{M_0} \cong \oplus_\pi \mathcal{A}(\pi),$$

where the sum is over all cuspidal automorphic representations $\pi \cong \sigma_1 \otimes \ldots \otimes \sigma_n$ of the Levi factor $M_0(\mathbb{A})$ such that for the group SO_{4n} all σ_i are isomorphic and have the trivial central character, while for the group Sp_{4n} all σ_i are isomorphic, have the trivial central character and $L(1/2, \sigma_i) \neq 0$. The irreducible space of automorphic forms $\mathcal{A}(\pi)$ is isomorphic to the image of the normalized intertwining operator

$$N(\underline{t}_0, \pi, w_{l,M_0})$$

where

$$\underline{t}_0 = (n-1/2, \ldots, 3/2, 1/2) \in \mathfrak{a}^*_{M_0,\mathbb{C}}$$

and w_{l,M_0} is the longest element of the Weyl group modulo M_0.

The part $L^2_T(SO_4)$ of the residual spectrum of $SO_{4n}(\mathbb{A})$ decomposes into

$$L^2_T(SO_4) \cong \oplus_\tau \mathcal{A}(\tau),$$

where the sum is over all cuspidal automorphic representations $\tau \cong \chi_1 \otimes \ldots \otimes \chi_{2n}$ of the torus $T(\mathbb{A})$ such that all χ_i are equal and χ_i^2 is trivial. The irreducible space of automorphic forms $\mathcal{A}(\tau)$ is isomorphic to the image of the normalized intertwining operator

$$N(\underline{t}_0, \tau, w_{l,T}),$$

where

$$\underline{t}_0 = (2n-1, \ldots, 1, 0) \in \mathfrak{a}^*_{T,\mathbb{C}}.$$

and $w_{l,T}$ is the longest element of the Weyl group.

The part $L^2_T(Sp_{4n})$ of the residual spectrum of $Sp_{4n}(\mathbb{A})$ decomposes into

$$L^2_T(Sp_{4n}) \cong (\oplus_\tau \mathcal{A}(\tau)) \oplus \mathcal{A}(\mathbf{1}_{T(\mathbb{A})}),$$

where the sum is over all cuspidal automorphic representations $\tau \cong \chi_1 \otimes \ldots \otimes \chi_{2n}$ of the torus $T(\mathbb{A})$ such that all χ_i are equal, quadratic and nontrivial, while the character $\mathbf{1}_{T(\mathbb{A})}$ is just the trivial character of $T(\mathbb{A})$. For the nontrivial τ, the space of automorphic forms $\mathcal{A}(\tau)$ is isomorphic to the sum of the irreducible representations of the form

$$\otimes_v \Pi_v,$$

where in the notation of the proof of Theorem 2.2, Π_v is one of at most two irreducible components $\Pi_v^\pm$ of the image of the normalized intertwining operator

$$N(\underline{t}_0, \tau, w_{l,T}),$$

where

$$\underline{t}_0 = (2n-1, \ldots, 1, 0) \in \mathfrak{a}^*_{T,\mathbb{C}}$$

and $w_{l,T}$ is the longest element of the Weyl group, such that $\Pi_v = \Pi_v^+$ for almost all v and the product of all the signs equals 1. The irreducible space of automorphic forms $\mathcal{A}(\mathbf{1}_{T(\mathbb{A})})$ is isomorphic to the image of the normalized intertwining operator

$$N(\underline{t}_0, \mathbf{1}_{T(\mathbb{A})}, w_{l,T}),$$

where

$$\underline{t}_0 = (2n, \ldots, 1) \in \mathfrak{a}^*_{T,\mathbb{C}}$$

and $w_{l,T}$ is the longest element of the Weyl group.

Proof. The proof of this Theorem for the split groups goes along the same lines as the proof of Theorem 2.2 for their inner forms above except for the decomposition of $L^2_T(Sp_{4n})$ for a nontrivial τ. Therefore, we first comment the split global normalization factors appearing in the calculation and then explain the result in that exceptional case.

The normalization factors for the local intertwining operators are at all places defined using the Langlands–Shahidi method for the generic representations at split places as in Section 1.1. Therefore, the global normalization factors needed for the decomposition of $L^2_{M_0}$ are the same as for the groups G'_n and H'_n in case (i). For the torus instead of the complicated normalization factor (19) in the $GL'_1 \subset H'_1$ case, now we have just the split $GL_1 \subset SL_2$ case where

$$r(s, \chi, w) = \frac{L(s, \chi)}{L(1+s, \chi)\varepsilon(s, \chi)},$$

and instead of (17) in $GL'_1 \times GL'_1 \subset GL'_2$ case we have the split $GL_1 \times GL_1 \subset GL_2$ case where

$$r((s_1, s_2), \chi_1 \otimes \chi_2, w) = \frac{L(s_1 - s_2, \chi_1\chi_2^{-1})}{L(1 + s_1 - s_2, \chi_1\chi_2^{-1})\varepsilon(s_1 - s_2, \chi_1\chi_2^{-1})}.$$

Observe that for the group SO_{4n} all the simple reflections correspond to $GL_1 \times GL_1 \subset GL_2$ case and that is the reason of a simpler decomposition.

For the group Sp_{4n} and a nontrivial τ, the character χ is nontrivial and hence the global normalizing factor $r(s, \chi, w)$ is holomorphic and non–vanishing for $Re(s) \geqslant 0$. Thus, the hyperplane $2s_n = 1$ is not singular and besides the usual singular hyperplanes $s_i - s_{i+1} = 1$ for $i = 1, \ldots, n-1$ appearing for $\chi = \chi_i = \chi_{i+1}$, we need the singular hyperplane $s_{n-1} + s_n = 1$ occuring if and only if $\chi = \chi_{n-1} = \chi_n^{-1}$, i.e. χ^2 is trivial. The iterated pole at $\underline{t}_0 = (2n-1, \ldots, 1, 0)$ indeed occurs for the intertwining operators corresponding to the Weyl group elements $w_{l,T}$ and w_1, where w_1 is as in the proof of Theorem 2.2. Since χ^2 is trivial, by the global functional equation, $r(s, \chi, w_0) = 1$

where $w_{l,T} = w_1 w_0$. Therefore, up to the nonzero constant, the iterated residue of the sum of the intertwining operators (20) equals

$$N(\underline{t}_0, \tau, w_1) + N(\underline{t}_0, \tau, w_{l,T}).$$

Decomposing according to the restricted tensor product over all places shows that the residue can be written as

$$N(\underline{t}_0, \tau, w_1)[Id + N(\underline{t}_0, \tau, w_0)],$$

where $N(\underline{t}_0, \tau, w_0)$ is in fact $SL_2(\mathbb{A})$ intertwining operator acting on the induced representation $\mathrm{Ind}_{GL_1(\mathbb{A})}^{SL_2(\mathbb{A})}\chi$. Now, the rest of the proof is the same as the end of the proof of Theorem 2.2. The parity condition on the product of the signs of the representations Π_v is just the non–vanishing condition for the term in square–brackets above. □

In the next Corollary we compare the parts of the residual spectrum of $G_n'(\mathbb{A})$ and $H_n'(\mathbb{A})$ obtained in Theorem 2.2 with the corresponding parts of the residual spectrum for the split groups $SO_{4n}(\mathbb{A})$ and $Sp_{4n}(\mathbb{A})$ obtained in Theorem 2.4. We use the notation of Theorems 2.2 and 2.4.

Corollary 2.5. *In case* (i) *let π be a cuspidal automorphic representation of the Levi factor $M_0(\mathbb{A})$ of the split group $SO_{4n}(\mathbb{A})$ or $Sp_{4n}(\mathbb{A})$ which is the global lift of π'. For one–dimensional π' in cases* (ii) *and* (iii) *let τ denote the one–dimensional cuspidal automorphic representation of the maximal split torus $T(\mathbb{A})$ of $SO_{4n}(\mathbb{A})$ or $Sp_{4n}(\mathbb{A})$ such that the global lift π is the unique irreducible quotient of the induced representation*

$$I_{T(\mathbb{A})}^{M_0(\mathbb{A})}((1/2, -1/2, \ldots, 1/2, -1/2), \tau).$$

Then the map

$$\imath : \mathcal{A}(\pi') \mapsto \begin{cases} \mathcal{A}(\pi), & \text{if } \pi' \text{ is in case (i)}, \\ \mathcal{A}(\tau), & \text{if } \pi' \text{ is in case (ii)}, \\ \mathcal{A}(\mathbf{1}_{T(\mathbb{A})}), & \text{if } \pi' \text{ is in case (iii)}, \end{cases}$$

is an injective map from the set of constituents $\mathcal{A}(\pi')$ of the part L^2 of the residual spectrum of $G_n'(\mathbb{A})$ or $H_n'(\mathbb{A})$ to the set of constituents of the part $L^2_{M_0} \oplus L^2_T(SO_{4n})$ or $L^2_{M_0} \oplus L^2_T(Sp_{4n})$ of the residual spectrum of the split group $SO_{4n}(\mathbb{A})$ or $Sp_{4n}(\mathbb{A})$. The image of the map $\imath$ consists of

- (a) *all constituents $\mathcal{A}(\pi)$ of $L^2_{M_0}$ such that π_v is square–integrable at every place $v \in S$, and*
- (b1) *for the group G_n', all constituents $\mathcal{A}(\tau)$ of $L^2_T(SO_{4n})$,*
- (b2) *for the group H_n', all constituents $\mathcal{A}(\tau)$ of $L^2_T(Sp_{4n})$ such that τ_v is nontrivial at every place $v \in S$ and the constituent $\mathcal{A}(\mathbf{1}_{T(\mathbb{A})})$.*

Proof. The Corollary is a direct consequence of Theorems 2.2 and 2.4. For the description of the image of $\imath$, let us just recall the global lift to $GL_2(\mathbb{A})$ of cuspidal automorphic representations of $GL_1'(\mathbb{A})$ defined at the beginning of Section 1. By Theorem (8.3) of [7] the global lift is a bijection between the cuspidal automorphic representations of $GL_1'(\mathbb{A})$ which are not one–dimensional and the cuspidal automorphic representations of $GL_2(\mathbb{A})$ having a square–integrable local component at all places $v \in S$. That gives condition in part (a) of the image of $\imath$. The global lift, as defined in Section 1 is also a bijection between the one–dimensional cuspidal automorphic representations of $GL_1'(\mathbb{A})$ and the residual automorphic representations of $GL_2(\mathbb{A})$. Hence, there is no reason for the conditions in (b1) and (b2) due to the global lift. However, the condition in (b2) is a consequence of the decomposition of L^2 for H_n' where the same local condition appears. The reason for such local condition lies in the fact that the local normalizing factors in the case $GL_1' \subset H_1'$ are not of the same form for split and non–split places. $\square$

Remark 2.6. *Observe that by our definition the map $\imath$ sends irreducible constituents to irreducible. But if $\mathcal{A}(\pi')$ is not irreducible, i.e. π' is in case* (ii) *for the group H_n', then $\imath(\mathcal{A}(\pi'))$ is also not irreducible. In Theorems 2.2 and 2.4 these spaces of automorphic forms are described more precisely. The choice of the local components Π_v' and Π_v at split places is the same, while at non–split places there is just one Π_v' and exactly two choices for Π_v since χ_v is nontrivial. Moreover, the product of all the signs of Π_v must be equal to one, thus reducing the freedom of the choice. Therefore, we can not refine the map $\imath$ to get matching of the irreducible constituents. The best we can do is to define $\imath(\otimes_v \Pi_v')$ to be the sum of all $\otimes_v \Pi_v$ such that $\Pi_v' \cong \Pi_v$ at all split places. Thus we obtained matching of the irreducible constituent of $\mathcal{A}(\pi')$ with the sum of $2^{|S|-1}$ irreducible constituents of $\imath(\mathcal{A}(\pi'))$.*

Finally, let F be a local field of characteristic zero and D_F the quaternion algebra central over F with the reduced norm $\det_F'$. The group of invertible elements of D_F is denoted by $GL_1'(F)$. Let $G_n'(F)$ and $H_n'(F)$ be the groups of isometries of the hermitian form on the $2n$–dimensional right vector space over D_F defined at the beginning of Section 1.

Now, we prove using the global method that the dual under the Aubert–Schneider–Stuhler involution defined in [4] and [34] of the principal series Steinberg representation of $G_n'(F)$ and $H_n'(F)$ is unitarizable. The proof is based on Theorem 2.2 where the parts of the residual spectrum of groups $G_n'(\mathbb{A})$ and $H_n'(\mathbb{A})$ for an arbitrary global quaternion algebra D central over an algebraic number field k are constructed. In fact, we show that the Aubert–Schneider–Stuhler dual of the principal series Steinberg representation is a local component of an automorphic representation belonging to the residual spectrum of $G_n'(\mathbb{A})$ or $H_n'(\mathbb{A})$ for a suitably chosen D and k, and thus unitarizable.

The principal series Steinberg representation of $G_n'(F)$ or $H_n'(F)$ is the Steinberg representation supported in the minimal parabolic subgroup with the Levi factor $M_0'(F) \cong GL_1'(F) \times \ldots \times GL_1'(F)$. For the group $G_n'(F)$ it is the unique irreducible subrepresentation of one of the induced representations

$$I\left((n-1/2,\ldots,3/2,1/2),\rho'\otimes\ldots\otimes\rho'\right)=\mathrm{Ind}_{M_0'(F)}^{G_n'(F)}\left(\rho'\nu^{n-1/2}\otimes\ldots\otimes\rho'\nu^{3/2}\otimes\rho'\nu^{1/2}\right),$$

where ρ' is not one–dimensional unitary irreducible representation of $GL_1'(F)$ with the trivial central character, and

$$I\left((2n-3/2,\ldots,5/2,1/2),(\mu\circ\det_F')\otimes\ldots\otimes(\mu\circ\det_F')\right)=$$
$$=\mathrm{Ind}_{M_0'(F)}^{G_n'(F)}\left((\mu\circ\det_F')\nu^{2n-3/2}\otimes\ldots\otimes(\mu\circ\det_F')\nu^{5/2}\otimes(\mu\circ\det_F')\nu^{1/2}\right),$$

where μ is a unitary character of $F^\times$, μ^2 is trivial and $\mu\circ\det_F'$ an one–dimensional unitary representation of $GL_1'(F)$. For the group H_n' it is the unique irreducible subrepresentation of one of the induced representations

$$I\left((n-1/2,\ldots,3/2,1/2),\rho'\otimes\ldots\otimes\rho'\right)=\mathrm{Ind}_{M_0'(F)}^{H_n'(F)}\left(\rho'\nu^{n-1/2}\otimes\ldots\otimes\rho'\nu^{3/2}\otimes\rho'\nu^{1/2}\right),$$

where ρ' is not one–dimensional unitary irreducible representation of $GL_1'(F)$ with the trivial central character,

$$I\left((2n-3/2,\ldots,5/2,1/2),(\mu\circ\det_F')\otimes\ldots\otimes(\mu\circ\det_F')\right)=$$

$$= \mathrm{Ind}_{M_0'(F)}^{H_n'(F)} \left((\mu \circ \det_F') \nu^{2n-3/2} \otimes \ldots \otimes (\mu \circ \det_F') \nu^{5/2} \otimes (\mu \circ \det_F') \nu^{1/2} \right),$$

where μ is a nontrivial unitary quadratic character of $F^\times$ and $\mu \circ \det_F'$ a nontrivial one–dimensional unitary representation of $GL_1'(F)$, and

$$I\left((2n-1/2, \ldots, 7/2, 3/2), (\mathbf{1}_F \circ \det_F') \otimes \ldots \otimes (\mathbf{1}_F \circ \det_F')\right) =$$
$$= \mathrm{Ind}_{M_0'(F)}^{H_n'(F)} \left((\mathbf{1}_F \circ \det_F') \nu^{2n-1/2} \otimes \ldots \otimes (\mathbf{1}_F \circ \det_F') \nu^{7/2} \otimes (\mathbf{1}_F \circ \det_F') \nu^{3/2} \right),$$

where $\mathbf{1}_F$ is the trivial character of $F^\times$ and $\mathbf{1}_F \circ \det_F'$ the trivial representation of $GL_1'(F)$. The Aubert–Schneider–Stuhler dual of these Steinberg representations is the unique irreducible quotient of the induced representations. It is in fact the Langlands quotient since the representations of $M_0'(F)$ are supercuspidal and all $\underline{s}$ are in the positive Weyl chamber.

Corollary 2.7. *The Aubert–Schneider–Stuhler dual of the principal series Steinberg representation of $G_n'(F)$ and $H_n'(F)$ is unitarizable, where for the group H_n' we assume that representation ρ' of $GL_1'(F)$ satisfies assumption $(*)$ of the proof below.*

Proof. Let k be an algebraic number field such that at a place w the completion k_w of k is isomorphic to F. Let D be a quaternion algebra central over k such that w is one of the places of k where D does not split, i.e. $w \in S$. Then $D \otimes_k k_w \cong D_F$.

For not one–dimensional unitary irreducible representation ρ' of $GL_1'(D_F) \cong D_F^\times$ let ρ be its local lift to $GL_2(F)$ defined using the Jacquet–Langlands correspondence at the beginning of Section 1. By Lemma 2.1 of [30] there exists a cuspidal automorphic representation $\sigma \cong \otimes_v \sigma_v$ of $GL_2(\mathbb{A})$ having the trivial central character and such that

$$\sigma_w \cong \rho.$$

For the group H_n' we assume that ρ' is such that there is a choice of σ satisfying the assumption

$$(*) \qquad L(1/2, \sigma) \neq 0,$$

for a suitable algebraic number field k. Let $\sigma' \cong \otimes_v \sigma_v'$ be the cuspidal automorphic representation of $GL_1'(\mathbb{A})$ with the global lift σ. Then

$$\sigma_w' \cong \rho'.$$

Similarly, for the nontrivial one–dimensional unitary representation $\mu \circ \det_F'$ of $GL_1'(F) \cong D_F^\times$ there exists a nontrivial unitary quadratic character χ of $\mathbb{A}^\times / k^\times$ such that

$$\chi_w \cong \mu$$

and χ_v is nontrivial at every place $v \in S$. For the trivial one–dimensional representation $\mathbf{1}_F \circ \det_F'$ of $GL_1'(F) \cong D_F^\times$, where $\mathbf{1}_F$ is the trivial character of $F^\times$, we take $\chi = \mathbf{1}$, where $\mathbf{1}$ is the trivial character of $\mathbb{A}^\times / k^\times$.

Now, by Theorem 2.2, for the representations $\pi' \cong \sigma' \otimes \ldots \otimes \sigma'$ and $\pi' \cong (\chi \circ \det') \otimes \ldots \otimes (\chi \circ \det')$ of the Levi factor $M_0'(\mathbb{A})$ and the corresponding $\underline{s}_0$ as in Theorem 2.2, the image of the normalized intertwining operator $N(\underline{s}_0, \pi', w_l)$ is isomorphic to a constituent of the residual spectrum of $G_n'(\mathbb{A})$ or $H_n'(\mathbb{A})$. Therefore, the image is unitary and specially at the place w of k the image of the local normalized intertwining operator

$$N(\underline{s}_0, \pi_w', w_l)$$

is unitary. But in Theorem 2.2 we have also proved that the image of that local normalized intertwining operator is irreducible. More precisely, for different π', it is precisely the Langlands quotient of the induced representations defining the Steinberg representations above. As mentioned there the Langlands quotients for those principal series representations are in fact the Aubert–Schneider–Stuhler duals of the Steinberg representation. □

References

[1] J. Arthur, Intertwining Operators and Residues I. Weighted Characters, *J. Funct. Anal.* **84** (1989), 19–84

[2] J. Arthur, An Introduction to the Trace Formula, in *Harmonic Analysis, the Trace Formula and Shimura Varieties*, *Clay Mathematics Proceedings* **4** (2005), 1–263

[3] J. Arthur, L. Clozel, *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula*, *Ann. of Math. Studies* **120**, Princeton Univ. Press, 1989

[4] A.–M. Aubert, Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longueur finie d'un groupe réductif p–adique, *Trans. Amer. Math. Soc.* **347** (1995), 2179–2189 (and Erratum, *Trans. Amer. Math. Soc.* **348** (1996), 4687–4690)

[5] W. Casselman, F. Shahidi, On Irreducibility of Standard Modules for Generic Representations, *Ann. Sci. École Norm. Sup.* **31** (1998), 561–589

[6] P. Deligne, D. Kazhdan, M.F. Vignéras, Représentations des algèbres centrales simples p–adiques, in *Représentations des Groupes Réductifs sur un Corps Local* (1984), Herman, Paris, 33–117

[7] S. Gelbart, H. Jacquet, Forms of $GL(2)$ from the Analytic Point of View, *Proc. Sympos. Pure Math.* **33**, part 1 (1979), 213–251

[8] N. Grbac, On the Residual Spectrum of Hermitian Quaternionic Inner Form of SO_8, preprint

[9] M. Hanzer, Unitary Dual of the Hermitian Quaternionic Group of the Split Rank 2, *Pacific J. Math*, to appear

[10] H. Jacquet, *Automorphic Forms on GL_2, Part II*, Lecture Notes in Math. **278**, Springer–Verlag, 1972

[11] H. Jacquet, Principal L–Functions of the Linear Group, *Proc. Sympos. Pure Math.* **33**, part 2 (1979), 63–86

[12] H. Jacquet, R.P. Langlands, *Automorphic Forms on GL_2*, Lecture Notes in Math. **114**, Springer–Verlag, 1970

[13] H. Jacquet, I.I. Piatetskii–Shapiro, J.A. Shalika, Rankin–Selberg Convolutions, *Amer. J. Math.* **105** (1983), 367–464

[14] H.H. Kim, The Residual Spectrum of Sp_4, *Compositio Math.* **99** (1995), 129–151

[15] H.H. Kim, The Residual Spectrum of G_2, *Canad. J. Math.* **48** (1996), 1245–1272

[16] H.H. Kim, Langlands–Shahidi Method and Poles of Automorphic L–functions: Application to Exterior Square L–Functions, *Canad. J. Math.* **51** (1999), 835–849

[17] H.H. Kim, Langlands–Shahidi Method and Poles of Automorphic L–functions II, *Israel J. Math.* **117** (2000), 261–284

[18] H.H. Kim, Residual Spectrum of Odd Orthogonal Groups, *Internat. Math. Res. Notices* **17** (2001), 873–906

[19] T. Kon–No, The Residual Spectrum of $U(2,2)$, *Trans. Amer. Math. Soc.* **350** (1998), 1285–1358

[20] R.P. Langlands, *On the Functional Equations Satisfied by Eisenstein series*, Lecture Notes in Math. **544**, Springer–Verlag, 1976

[21] C. Mœglin, Orbites unipotentes et spectre discret non ramifie, *Compositio Math.* **77** (1991), 1–54

[22] C. Mœglin, Représentations unipotentes et formes automorphes de carré intégrable, *Forum Math.* **6** (1994), 651–744

[23] C. Mœglin, Conjectures sur le spectre residuel, *J. Math. Soc. Japan* **53** (2001), 395–427

[24] C. Mœglin, J.–L. Waldspurger, Le spectre résiduel de $GL(n)$, *Ann. Sci. École Norm. Sup.* **22** (1989), 605–674

[25] C. Mœglin, J.–L. Waldspurger, *Spectral Decomposition and Eisenstein Series*, Cambridge Tracts in Math. **113**, Cambridge University Press, 1995

[26] G. Muić, The Unitary Dual of p–adic G_2, *Duke Math. J.* **90** (1997), 465–493

[27] G. Muić, Some Results on Square Integrable Representations; Irreducibility of Standard Representations, *Internat. Math. Res. Notices* **14** (1998), 705–726

[28] G. Muić, A Proof of Casselman–Shahidi's Conjecture for Quasi–split Classical Groups, *Canad. Math. Bull.* **44** (2001), 298–312

[29] G. Muić, On Certain Classes of Unitary Representations for Split Classical Groups, *Canad. J. Math.*, to appear

[30] G. Muić, G. Savin, Complementary Series for Hermitian Quaternionic Groups, *Canad. Math. Bull.* **43** (2000), 90–99

[31] F. Shahidi, On Certain L–Functions, *Amer. J. Math.* **103** (1981), 297–355

[32] F. Shahidi, A proof of Langlands' Conjecture on Plancherel Measures; Complementary series of $\mathfrak{p}$–adic Groups, *Ann. of Math.* **132** (1990), 273–330

[33] B. Speh, Unitary Representations of $Gl(n,\mathbb{R})$ with nontrivial $(\mathfrak{g},K)$–cohomology, *Invent. Math.* **71** (1983), 443–465

[34] P. Schneider, U. Stuhler, Representation Theory and Sheaves on the Bruhat–Tits Building, *Publ. Math. IHES* **85** (1997), 97–191

[35] M. Tadić, Classification of Unitary Representations in Irreducible Representations of General Linear Group (Non–Archimedean Case), *Ann. Sci. École Norm. Sup.* **19** (1986), 335–382

[36] M. Tadić, Induced Representations of $GL(n,A)$ for p–adic Division Algebras A, *J. Reine Angew. Math.* **405** (1990), 48–77

[37] J. Tate, Fourier Analysis in Number Fields and Hecke's Zeta–Functions, Harvard Dissertation, 1950, in *Algebraic Number Theory*, Academic Press, Boston, 1967

[38] D. Vogan, Gelfand–Kirillov Dimension for Harish–Chandra Modules, *Invent. Math.* **48** (1978), 75–98

[39] Y. Zhang, The Holomorphy and Nonvanishing of Normalized Local Intertwining Operators, *Pacific J. Math.* **180** (1997), 385–398

[40] S. Žampera, The Residual Spectrum of the Group of Type G_2, *J. Math. Pures Appl.* **76** (1997), 805–835

Neven Grbac, Department of Mathematics, University of Zagreb, Unska 3, 10000 Zagreb, Croatia
E-mail address: `neven.grbac@zpm.fer.hr`