On Hall's conjecture

(Talk)

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Hall's conjecture asserts that for any $\varepsilon > 0$, there exists a constant $c(\varepsilon) > 0$ such that if x and y are positive integers satisfying $x^3 - y^2 \neq 0$, then $|x^3 - y^2| > c(\varepsilon)x^{1/2-\varepsilon}$. It is known that Hall's conjecture follows from the *abc*-conjecture. Danilov proved that $0 < |x^3 - y^2| < 0.97\sqrt{x}$ has infinitely many solutions in positive integers x, y.

Davenport proved that for non-constant complex polynomials x and y, such that $x^3 \neq y^2$, the inequality

$$\deg(x^3 - y^2) \geq \tfrac{1}{2}\deg(x) + 1 \qquad (1)$$

holds. This statement also follows from Stothers-Mason's *abc* theorem for polynomials. Zannier proved that for any positive integer δ there exist complex polynomials x and y such that $\deg(x) = 2\delta$, $\deg(y) = 3\delta$ and x, y satisfy the equality in Davenport's bound (1).

It is natural to ask whether examples with the equality in (1) exist for polynomials with integer (rational) coefficients. Such examples are known only for $\delta = 1, 2, 3, 4, 5$. The examples for $\delta = 5$ were found by Birch, Chowla, Hall, Schinzel and Elkies. In these examples we have $\deg(x^3 - y^2)/\deg(x) = 0.6$, and it seems that no examples of polynomials with integer coefficients, satisfying $x^3 - y^2 \neq 0$ and $\deg(x^3 - y^2)/\deg(x) < 0.6$, were known.

In this talk we will present our recent result which gives an explicit construction of integer polynomials x and y of arbitrarily large degrees with $\deg(x^3 - y^2) - \frac{1}{2}\deg(x)$ bounded from above by an absolute constant.

For any $\varepsilon > 0$ there exist polynomials x and y with integer coefficients such that $x^3 \neq y^2$ and $\deg(x^3 - y^2)/\deg(x) < 1/2 + \varepsilon$. More precisely, for any even positive integer δ there exist polynomials x and y with integer coefficients such that $\deg(x) = 2\delta$, $\deg(y) = 3\delta$ and $\deg(x^3 - y^2) = \delta + 5$.

The construction is based on the binary recursive sequence of polynomials given by

$$a_1 = 0, \quad a_2 = t^2 + 1, \quad a_m = 2ta_{m-1} + a_{m-2}.$$

We give also an interpretation of our construction in terms of solutions of polynomial Pell's equation $X^2 - (t^2 + 1)Y^2 = -1$.

MSC2010: 11C08, 11D25, 11D75.

Keywords: Hall's conjecture, integer polynomials.

Section: Number Theory.