Multi-Structured Designs and Their Applications

Ryoh Fuji-Hara University of Tsukuba, Japan Joint work with Ying Miao (University of Tsukuba) Block Design (V, B)

- V: a finite set (points)
- \mathfrak{B} : a collection of subsets (blocks) of V

$$\mathfrak{B} = \{\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_b\} \ , \ \mathbf{B}_i \subseteq V$$

some combinatorial conditions

Conditions

- (CI) every block contains k points (regular)
- (C2) every point of V is contained in *r* blocks (singleton balance)
- (C3) every pair of distinct elements of V appears in exactly λ blocks (pair balance)

Classical designs

Balanced Incomplete Block Design (BIBD) or 2-design: when (C1), (C2) and (C3) are

Multi-Structured Design

- A block design
- Each block has further structure
- additional combinatorial conditions

Block Forms

Each block \mathbf{B}_i has sub-blocks

$$\mathbf{B}_i = \{C_{i1}, C_{i2}, \dots, C_{in_i}\}, \ C_{ij} \subseteq \mathbf{B}_i$$

)
$$\{C_{i1}, C_{i2}, ..., C_{in_i}\}$$
 is a partition of \mathbf{B}_i

II)
$$C_{ij} \subseteq \mathbf{B}_i$$
 and $\mathbf{B}_i = \bigcup_{1 \le j \le n_i} C_{ij}$

Unordered

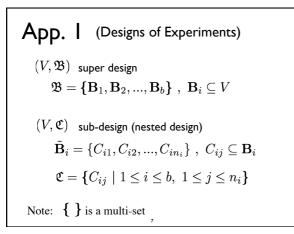
Each block \mathbf{B}_i is a set of disjoint n_i sub-blocks

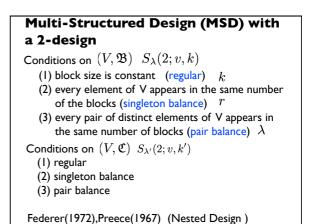
$$\tilde{\mathbf{B}}_i = \{C_{i1}, C_{i2}, ..., C_{in_i}\} \ , \ C_{ij} \subseteq \mathbf{B}_i$$

Ordered

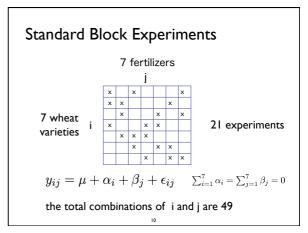
Each block \mathbf{B}_i is an ordered set of disjoint n subblocks (some of them can be empty)

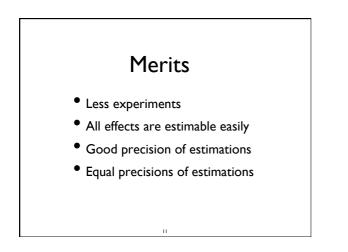
$$\vec{\mathbf{B}}_i = (C_{i1}, C_{i2}, ..., C_{in}) \ , \ C_{ij} \subseteq \mathbf{B}_i$$

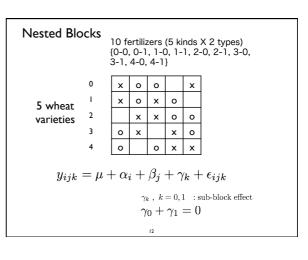


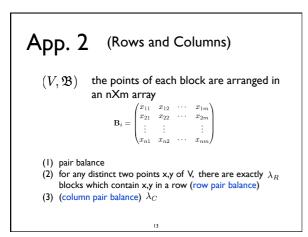


Example $V=\{0,1,2,3,4\} S_3(2;5,4)$ $B_1=\{ \{0, 1\}, \{2, 4\} \}$ $B_2=\{ \{1, 2\}, \{3, 0\} \}$ $B_3=\{ \{2, 3\}, \{4, 1\} \}$ $B_4=\{ \{3, 4\}, \{0, 2\} \}$ $B_5=\{ \{4, 0\}, \{1, 3\} \}$

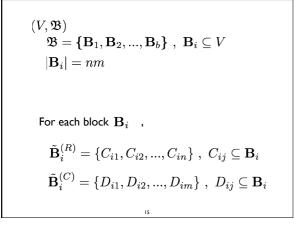




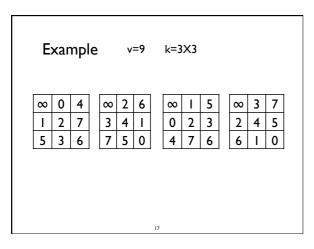


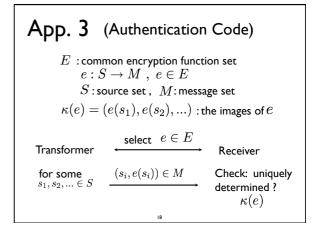


Experiments Srivastava (1978) Singh and Dey (1978) $y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + \delta_l + \epsilon_{ijkl}$ α_i : variety effect β_j : block effect γ_k : row effect δ_l : column effect



Orthogonal MSDs								
Two "MSD with a 2-design"s with the conditions								
I. (V, \mathfrak{C}) $\mathfrak{C} = \{C_{ij} \mid 1 \le i \le b, \ 1 \le j \le n\}$ (1) regular (2) pair balance								
2. (V, \mathfrak{D}) $\mathfrak{D} = \{D_{ij} \mid 1 \leq i \leq b, \ 1 \leq j \leq m\}$ (1) regular (2) pair balance								
3. For each i , $ C_{ij} \cap D_{ik} = 1$, $1 \leq j \leq n$, $1 \leq k \leq m$								





sources							
		S 0	SI	S 2			
	e0	0	0	0			
	eı	Ι	Ι	2			
	e ₂	2	2	Ι	There are no		
encryption functions	e3	0			two images		
TUTICUOTIS	e4	Ι	2	0	including $e_i(s_j), e_i(s_k), j \neq k$		
	es	2	0	2			
	e6	0	2	2			
	e7		0				
	e8	2		0	M= {0,1,2}		
		_	19				

MSD with External Packing

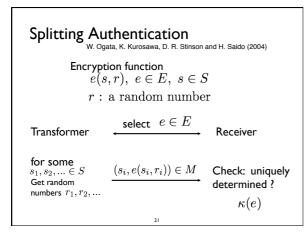
Each block \mathbf{B}_i consists of a set of disjoint n_i sub-blocks

$$\mathbf{\tilde{B}}_{i} = \{C_{i1}, C_{i2}, \dots, C_{in_{i}}\}, \ C_{ij} \subseteq \mathbf{B}_{i}$$

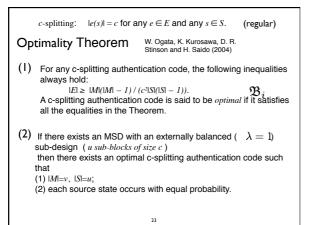
Conditions

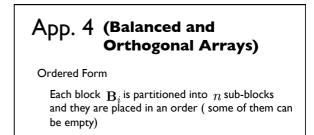
(1) For any $x, y \in V$, there is at most one super block which contains x, y in distinct sub-blocks

(2)
$$|C_{ij}| \ge 1, \ n_i = n$$
, for any $i = 1, 2, ..., b$



	S0	SI	S 2					
e ₀	0, I	2, 4	12,20					
eı	7, 8	9, 11	19, 2					
e ₂	5,6	7, 9	17,0					
e3	3, 4	5, 7	15	There are no two				
e4	9, 10	11	21,4	rows which contain				
es	2, 3	4, 6	14, 22	$e_i(s_i, r), e_i(s_k, r'), j \neq k$				
e6	10, 11	12, 14	5					
e7	4, 5	6, 8	16	in different columns				
e8	6, 7	8, 10	18, 1					
e 9	١, 2	3, 5	13,21					
e10	8, 9	10, 12	20, 3					
eii	11,12	13, 15	23,6					
M={0,1,2,,24}								

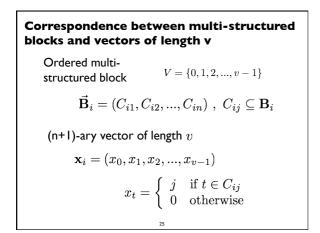




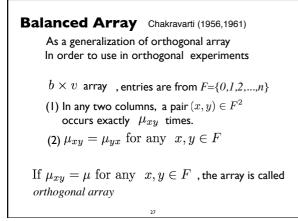
$$\vec{\mathbf{B}}_i = (C_{i1}, C_{i2}, ..., C_{in}) \ , \ C_{ij} \subseteq \mathbf{B}_i$$

 $\mathfrak{C}_j\;$: the set of j-th sub-blocks $\;\;1\leq j\leq n\;$

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Example $V = \{0, 1, 2,, 6\}$	
$ec{\mathbf{B}}_1 = (\{0,1\},\{2,5\})$ $ec{\mathbf{B}}_2 = (\{0,4\},\{3,6\})$	$\mathbf{x}_1 = (1, 1, 2, 0, 0, 2, 0)$ $\mathbf{x}_2 = (1, 0, 0, 2, 1, 0, 2)$
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I	I	2	1	0	0	0	п		0	2	0	2	0
2	I	Т	0	0	I	0	12	2	0	Т	0	2	0
	2	0	2	0	0	Т	13	Т	0	0	Т	0	I
ł	2	0	0	0	Т	2	14	Т	0	0	2	0	2
	0	2	2	2	0	0	15	0	Т	T	0	0	Ι
	0	Ι	0	Ι	2	0	16	0	Т	2	0	0	2
,	0	0	Т	Т	0	2	17	0	2	0	0	Т	Ι
3	0	0	0	2	2	Ι	18	0	2	0	0	2	2
,	2	Т	0	2	0	0	19	0	0	Т	2	Т	0
0	2	2	0	Т	0	0	20	0	0	2	Т	Т	0

Ordered MSD with Mutually Balanced (r,λ) -designs

- $\begin{array}{ll} [{\bf I}] & (V, \mathfrak{B}) & & ({\bf I}) \text{ singleton balance } (r) \\ & & ({\bf 2}) \text{ pair balance } (\lambda) \end{array}$
- $\begin{array}{ll} \mbox{[2]} & (V, \mathfrak{C}_j) & ({\rm I}) \mbox{ singleton balance } (r_j) \\ & (2) \mbox{ pair balance } (\lambda_j) \\ & \mbox{ for } j {=} {\rm I}, 2, ..., {\rm n} \end{array}$
- [3] External Balance : For any distinct $x, y \in V$, there are exactly μ_{ij} super blocks, each of which contains x in the *i*-th and y in the *j*-th sub-block.

Let
$$\mathfrak{C}_0 = \{V \setminus \mathbf{B}_i \mid 1 \le i \le b\}$$

Lemma

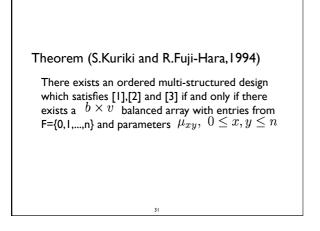
If the conditions [1] and [2] are satisfied then (V, \mathfrak{C}_0) holds singleton and pair balance property $r_{\bullet} = b - r_{\bullet}$

$$r_0 = b - r$$
$$\lambda_0 = b - 2r + \lambda$$

Lemma
Further, if the condition [3] is satisfied for i,j = 1,2,...n then it works for
$$0 \le i, j \le n$$

$$\mu_{i0} = \mu_{0i} = r_i - \sum_{j=1}^n \mu_{ij}$$

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Example V={1,2,3,4,	5,6}
$\begin{array}{ccccc} (1 , 3 & 2 &) \\ (1 , 2 , 5 & \emptyset &) \\ (6 & 1 , 3 &) \\ (5 & 1 , 6 &) \\ (\emptyset & 2 , 3 , 4) \\ (2 , 4 & 5 &) \\ (3 , 4 & 6 &) \\ (6 & 4 , 5 &) \\ (2 & 1 , 4 &) \\ (4 & 1 , 2 &) \\ (1 & 3 , 5 &) \end{array}$	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$
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