Ime i prezime:

1.(20)	2.(20)	3.(20)	4.(20)	5. (20)	$\sum$

KOMPLEKSNA ANALIZA

Popravni ispit, 02.07.2013.

1. Riješite jednadžbu:

$$e^{6z} + e^{3z+5} - e^{3z-5} - 1 = 0.$$

Rj. $t = e^{3z}$

$$(e^{3z})^2 + e^5 \cdot e^{3z} - e^{3z} \cdot e^{-5} - 1 = 0$$

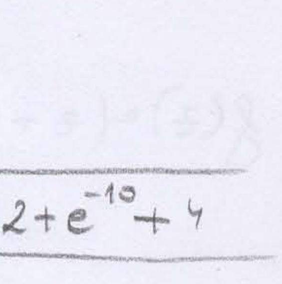
$$t^2 + t(e^5 - e^{-5}) - 1 = 0$$

$$t_{12} = \frac{-(e^5 - e^{-5}) \pm \sqrt{e^{10} - 2 + e^{-10} + 4}}{2}$$

$$= \frac{-e^5 + e^{-5} \pm \sqrt{(e^5 + e^{-5})^2}}{2}$$

$$= \frac{-e^5 + e^{-5} \pm (e^5 + e^{-5})}{2}$$

$$t_1 = \underset{>0}{e^{-5}}, \quad t_2 = -e^5$$



$$z_1 = \text{Ln}\, t_1 = \ln|e^{-5}| + i \cdot (\arg(e^{-5}) + 2k\pi)$$

$$= \ln e^{-5} + i \cdot (0 + 2k\pi)$$

$$= -5 + 2k\pi i, \quad k \in \mathbb{Z}$$

$$z_2 = \text{Ln}\, t_2 = \ln|-e^5| + i \cdot (\underbrace{\arg(-e^5)}_{\pi} + 2k\pi)$$

$$= 5 + i \cdot \pi(2k+1), \quad k \in \mathbb{Z}$$

2. Odredite analitičku funkciju $f = u + iv$ ako je njen imaginarni dio jednak

$$v(x, y) = e^{-y}((x+1)\sin x + y\cos x) + y$$

te vrijedi $f(0) = 0$.

RJ. $\frac{\partial v}{\partial x} = e^{-y}(\sin x + (x+1)\cos x - y\sin x)$

$\frac{\partial v}{\partial y} = -e^{-y}((x+1)\sin x + y\cos x) + e^{-y} \cdot \cos x + 1$

$= \frac{\partial u}{\partial x} \quad / \int dx$

$u(x,y) = -e^{-y}\int (x+1)\sin x - y e^{-y}\int \cos x + e^{-y}\int \cos x + x + \varphi(y)$

$\left\{ \begin{array}{l} u = x+1 \Rightarrow du = dx \\ dv = \sin x dx \Rightarrow v = -\cos x \end{array} \right\}$

$= -e^{-y}\left(-(x+1)\cos x + \underbrace{\int \cos x \, dx}_{=\sin x}\right) - y e^{-y}\sin x + e^{-y}\sin x + x + \varphi(y)$

$= e^{-y}(x+1)\cos x - \cancel{e^{-y}\sin x} - y e^{-y}\sin x + \cancel{e^{-y}\sin x} + x + \varphi(y) \quad / \frac{\partial}{\partial y}$

$\frac{\partial u}{\partial y} = -e^{-y}\cancel{(x+1)\cos x} - \cancel{e^{-y}\sin x} + \cancel{y e^{-y}}\sin x + \varphi'(y)$

$= -\cancel{e^{-y}\sin x} - \cancel{e^{-y}(x+1)}\cos x + \cancel{y e^{-y}}\sin x$

$\Rightarrow \varphi'(y) = 0 \Rightarrow \varphi(y) = C$

$f(z) = e^{-y}((x+1)\cos x - y\sin x) + i \cdot e^{-y}((x+1)\sin x + y\cos x)$

$+ x + iy + C$

$= e^{-y}(x+1)(\cos x + i\sin x) + e^{-y} y (i\cos x - \sin x)$

$+ z + C$

$$= e^{-y} \cdot (x+1) e^{ix} + i e^{-y} y e^{ix} + z + C$$

$$= \underbrace{e^{-y} \cdot e^{ix}}_{e^{i(\underbrace{x+iy}_{z})}} \left(\underbrace{(x+1)+iy}_{z+1}\right) + z + C$$

$$= (z+1) \cdot e^{iz} + z + C$$

$$f(0) = 0 \iff (0+1) \cdot \underbrace{e^{0}}_{1} + 0 + C = 0$$

$$1 + C = 0$$

$$\iff C = -1$$

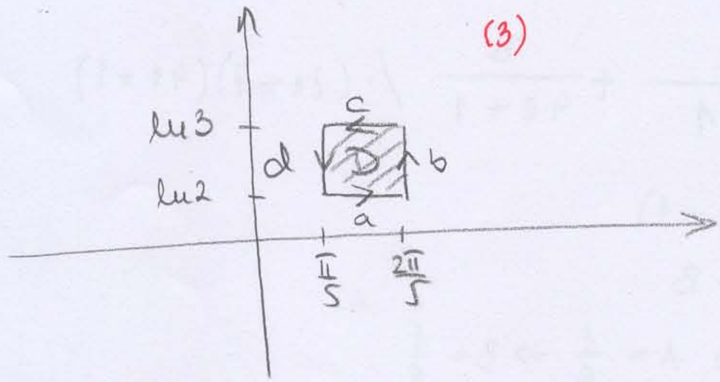
$$\Rightarrow f(z) = (z+1) \cdot e^{iz} + z - 1$$

3. Odredite sliku područja

$$D = \{z \in \mathbb{C} : \frac{\pi}{5} < \text{Re}z < \frac{2\pi}{5}, \ln 2 < \text{Im}z < \ln 3\}$$

pri preslikavanju $w = e^{i(z-\pi)}$. Skicirajte D i $w(D)$.

RJ.



(3)

$$w = e^{i(z-\pi)} = e^{i(x+iy-\pi)} = e^{ix-y-i\pi} = e^{-y+i(x-\pi)} = e^{-y}(\cos(x-\pi)+i\sin(x-\pi))$$

$\to |w| = e^{-y}$, $\arg w = x-\pi$ (5)

$a \ldots \frac{\pi}{5} \leq x \leq \frac{2\pi}{5}$, $y = \ln 2$ (2)

$a^* \ldots |w| = e^{-y} = e^{-\ln 2} = e^{\ln 2^{-1}} = \frac{1}{2}$

$\arg w \leq \frac{2\pi}{5} - \pi \Rightarrow \arg w \leq -\frac{3\pi}{5}$

$\arg w \geq \frac{\pi}{5} - \pi \Rightarrow \arg w \geq -\frac{4\pi}{5}$ (2)

$b \ldots x = \frac{2\pi}{5}$, $\ln 2 \leq y \leq \ln 3$ (2)

$b^* \ldots |w| \in [e^{-\ln 2}, e^{-\ln 3}] = [\frac{1}{2}, \frac{1}{3}]$ (2)

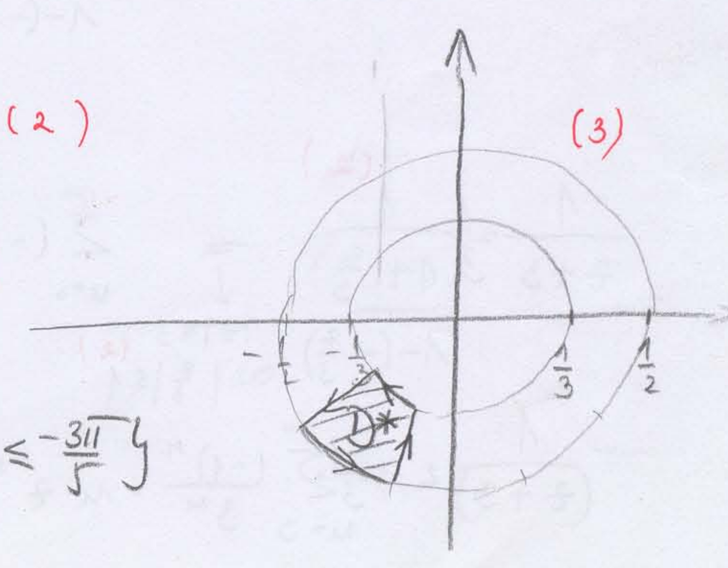
$\arg w = \frac{2\pi}{5} - \pi = -\frac{3\pi}{5}$

$c \ldots x \in [\frac{2\pi}{5}, \frac{\pi}{5}]$, $y = \ln 3$ (2)

$c^* \ldots |w| = \frac{1}{3}$, $\arg w \in [-\frac{3\pi}{5}, -\frac{4\pi}{5}]$ (2)

$d \ldots x = \frac{\pi}{5}$, $y \in [\ln 3, \ln 2]$ (2)

$d^* \ldots |w| \in [\frac{1}{3}, \frac{1}{2}]$, $\arg w = -\frac{4\pi}{5}$ (2)



(3)

$$D^* = \{w \in \mathbb{C} : |\frac{1}{3}| < |w| < \frac{1}{2}, -\frac{4\pi}{5} \leq \arg w \leq -\frac{3\pi}{5}\}$$

4. Razvijte u Laurentov red oko 0 funkciju

$$f(z) = \frac{z}{(2z-1)(4z+1)} + \frac{3z}{(z+3)^2}$$

na području $D = \{z \in \mathbb{C} : \frac{1}{2} \leqslant |z| \leqslant 3\}$.

RJ. $\frac{z}{(2z-1)(4z+1)} = \frac{A}{2z-1} + \frac{B}{4z+1} \quad / \cdot (2z-1)(4z+1)$

$z = A(4z+1) + B(2z-1)$

$= z(4A+2B) + A - B$

$\Rightarrow 4A + 2B = 1 \quad \Rightarrow A = \frac{1}{6} \Rightarrow B = \frac{1}{6}$

$A - B = 0 \Rightarrow B = A$ (2)

$\Rightarrow f(z) = \frac{1}{6} \cdot \underbrace{\frac{1}{2z-1}}_{f_1(z)} + \frac{1}{6} \cdot \underbrace{\frac{1}{4z+1}}_{f_2(z)} + \underbrace{\frac{3z}{(z+3)^2}}_{f_3(z)}$

$f_1(z) = \frac{1}{2z-1} = \frac{1}{2z} \cdot \frac{1}{1 - \frac{1}{2z}} \overset{(2)}{=} \frac{1}{2z} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n$

$|z| \geq \frac{1}{2}$ (2) $\Leftrightarrow \left|\frac{1}{2z}\right| \leq 1$

$= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot \frac{1}{z^{n+1}}$ (2)

$f_2(z) = \frac{1}{4z+1} = \frac{1}{4z} \cdot \frac{1}{\underbrace{1 + \frac{1}{4z}}_{1-(-\frac{1}{4z})}}$ (2) $= \frac{1}{4z} \cdot \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{4z}\right)^n$

$|z| \geq \frac{1}{2}$ (2)

$\Rightarrow |z| \geq \frac{1}{4}$

$\Rightarrow \left|\frac{1}{4z}\right| \leq 1$

$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{4^{n+1}} \cdot \frac{1}{z^{n+1}}$ (2)

$\frac{1}{z+3} = \frac{1}{3} \cdot \frac{1}{\underbrace{1 + \frac{z}{3}}_{1-(-\frac{z}{3})}}$ (2) $= \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} z^n \; /'$ (2)

$|z| \leq 3$

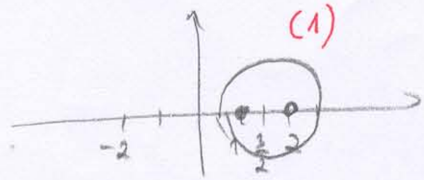
$\Leftrightarrow \left|\frac{z}{3}\right| \leq 1$ (2)

$-\frac{1}{(z+3)^2} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} \cdot n z^{n-1} \Rightarrow f_3(z) = \frac{3z}{(z+3)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{3^n} \cdot n \cdot z^{n-1}$

5. Izračunajte:

$$\int_{|z-\frac{3}{2}|=1} \left(\frac{e^{z^3}}{(z-1)^2(z+2)} + (z^2+z+1)\operatorname{ch}\frac{1}{z-2} \right) dz.$$

(1)



RJ. $I_1 = \int_{|z-\frac{3}{2}|=1} \underbrace{\frac{e^{z^3}}{(z-1)^2(z+2)}}_{g(z)} dz$

(2)

g je analitička na $|z-\frac{3}{2}|\leq 1$, osim u 1

$\rightarrow I_1 = \int_{|z-\frac{3}{2}|=1} \frac{g(z)}{(z-1)^2} \underset{\text{C-int. formula}}{=} g'(1) \cdot 2\pi i$ (1)

$g(z) = \frac{e^{z^3}}{(z+2)}$

$g'(z) = \frac{e^{z^3} \cdot 3z^2 \cdot (z+2) + e^{z^3}}{(z+2)^2}$ (2)

$\rightarrow g'(1) = \frac{e \cdot 3 \cdot 3 + e}{9} = \frac{10}{9} e$ (1)

$\Rightarrow I_1 = 2\pi i \cdot \frac{10}{9} e$

$I_2 = \int_{|z-\frac{3}{2}|=1} \underbrace{(z^2+z+1) \operatorname{ch} \frac{1}{z-2}}_{f(z)} = 2\pi i \cdot \operatorname{Res} f(2)$ (1)

$\mapsto$ singularitet: $z=2$ (bitni)

$f(z) = (z^2+z+1)\operatorname{ch}\frac{1}{z-2} = \left\{ \begin{array}{l} w = z-2 \\ z = w+2 \end{array} \right\}$

$= ((w+2)^2 + w + 2 + 1) \operatorname{ch} \frac{1}{w}$

$= (w^2 + 4w + 4 + w + 3) \sum_{n=0}^{\infty} \left(\frac{1}{w}\right)^{2n} \cdot \frac{1}{(2n)!}$

$= (w^2 + 5w + 7) \sum_{n=0}^{\infty} \frac{1}{(2n)!} \cdot \left(\frac{1}{w}\right)^{2n}$ (12)

$\rightarrow \operatorname{Res} f(2) = 5 \cdot \frac{1}{2} = \frac{5}{2}$

$\rightarrow I_2 = 5\pi i$

$\rightarrow I = I_1 + I_2 = \pi i \left(\frac{20}{9} e + 5 \right)$