Ime i prezime:

1.(20)	2.(20)	3.(20)	4.(20)	5. (20)	Σ

KOMPLEKSNA ANALIZA

Popravni ispit, 18.06.2013.

1. Riješite jednadžbu:

$$e^{2z} + 2e^{z} - 3 = 0.$$

Rj. $t = e^{z}$

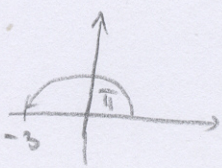
$$t^2 + 2t - 3 = 0$$

$$t_{12} = \frac{-2 \pm \sqrt{4+12}}{2} = \frac{-2 \pm 4}{2}$$

$t_1 = -3$

$t_2 = 1$

$$e^{z_1} = -3$$



$$\Rightarrow z_1 = \mathrm{Ln}(-3)$$

$$= \ln|-3| + i(\underbrace{\arg(-3)}_{=\pi} + 2k\pi)$$

$$= \ln 3 + i\pi(2k+1), \quad k \in \mathbb{Z}$$

$$z_2 = \mathrm{Ln}(1)$$

$$= \ln|1| + i(\underbrace{\arg(1)}_{=0} + 2k\pi)$$

$$= \underbrace{\ln 1}_{=0} + 2k\pi i$$

$$= 2k\pi i, \quad k \in \mathbb{Z}$$

2. Odredite analitičku funkciju $f = u + iv$ ako je njen realni dio jednak

$$u(x,y) = e^x(x\cos y - y\sin y)$$

te vrijedi $f(0) = 0$.

Rj. $\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x \cdot \cos y = e^x((x+1)\cos y - y\sin y)$

$\Rightarrow \frac{\partial v}{\partial y} = e^x((x+1)\cos y - y\sin y) \quad / \int dy$

$v(x,y) = e^x(x+1)\int \cos y\,dy - e^x\int y\sin y + \varphi(x) \qquad = \left[\begin{array}{l} u=y \Rightarrow du=d \\ dv=\sin y\,dy \Rightarrow v=-\cos \end{array}\right.$

$= e^x(x+1)\sin y - e^x(-y\cos y + \int\cos y\,dy) + \varphi(x)$

$= e^x(x+1)\sin y + e^x y\cos y - e^x\sin y + \varphi(x)$

$= e^x x\sin y + e^x y\cos y + \varphi(x) \quad / \frac{d}{dx}$

$\Rightarrow \frac{\partial v}{\partial x} = e^x x\sin y + e^x\sin y + e^x y\cos y + \varphi'(x)$

$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y)$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \iff e^x x\sin y + e^x\cancel{\sin y} + e^x\cancel{y\cos y} + \varphi'(x) = e^x\cancel{x\sin y} + e^x\cancel{\sin y} + \cancel{e^x y\cos y}$

$\varphi'(x) = 0 \iff \varphi(x) = C$

$\Rightarrow v(x,y) = e^x x\sin y + e^x y\cos y + C$

$f(z) = u + iv = e^x x\cos y - e^x y\sin y + ie^x x\sin y + ie^x y\cos y + iC$

$= xe^x\underbrace{(\cos y + i\sin y)}_{=e^{iy}} + \underbrace{e^x y(i\cos y - \sin y)}_{=ie^x y\underbrace{(\cos y + i\sin y)}_{e^{iy}}} + iC$

$= e^{iy}(x+iy)e^x + iC$

$= e^{x+iy}(x+iy) + iC$

$= ze^z + iC$

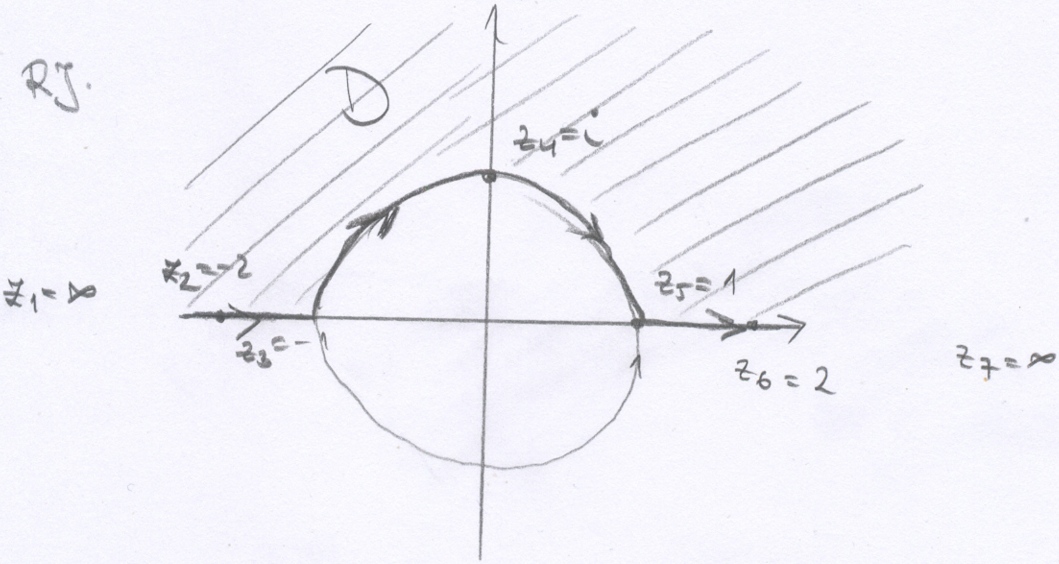
$f(0) = 0 \Rightarrow iC = 0 \Rightarrow C = 0 \Rightarrow f(z) = ze^z$

3. Odredite sliku područja

$$D = \{z \in \mathbb{C} : |z| > 1, \operatorname{Im} z > 0\}$$

pri preslikavanju $w = \dfrac{z}{z-i}$. Skicirajte D i $w(D)$.

RJ.



$$z_1 = \infty \Rightarrow w_1 = \lim_{z\to\infty} \frac{z}{z-i} = 1$$

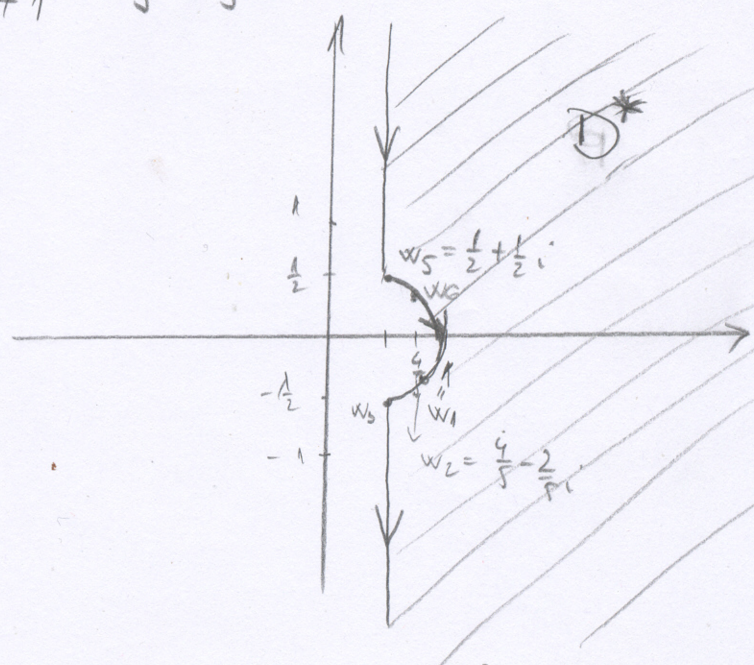
$$z_2 = -2 \Rightarrow w_2 = \frac{-2}{-2-i} \cdot \frac{-2+i}{-2+i} = \frac{4-2i}{4+1} = \frac{4}{5} - \frac{2}{5}i$$

$$z_3 = -1 \Rightarrow w_3 = \frac{1}{-1-i} \cdot \frac{-1+i}{-1+i} = \frac{-1+i}{1+1} = -\frac{1}{2} + \frac{1}{2}i$$

$$z_4 = i \Rightarrow w_4 = \frac{i}{i-i} = \infty$$

$$z_5 = 1 \Rightarrow w_5 = \frac{1}{1-i} \cdot \frac{1+i}{1+i} = \frac{1+i}{1+1} = \frac{1}{2} + \frac{1}{2}i$$

$$z_6 = 2 \Rightarrow w_6 = \frac{2}{2-i} \cdot \frac{2+i}{2+i} = \frac{4+2i}{4+1} = \frac{4}{5} + \frac{2}{5}i$$



4. Razvijte u Laurentov red oko 0 funkciju

$$f(z) = \frac{z^2 - 2z + 7}{(4 + z^2)(3 - 2z)}$$

na području $D = \{z \in \mathbb{C} : \frac{3}{2} \leq |z| \leq 2\}$.

RJ. $$f(z) = \frac{z^2 - 2z + 7}{(4+z^2)(3-2z)} = \frac{z^2 - 2z + 4 + 3}{(4+z^2)(3-2z)} = \underbrace{\frac{1}{3-2z}}_{f_1(z)} + \underbrace{\frac{1}{4+z^2}}_{f_2(z)}$$

$$|z| \geq \frac{3}{2} \Leftrightarrow \left|\frac{3}{2z}\right| \leq 1$$

$$f_1(z) = \frac{1}{3-2z} = \frac{1}{-2z} \cdot \frac{1}{1 - \frac{3}{2z}} = -\frac{1}{2z} \cdot \sum_{n=0}^{\infty} \left(\frac{3}{2z}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{-3^n}{2^{n+1}} \cdot \frac{1}{z^n}$$

$$|z| \leq 2 \Leftrightarrow \left|\frac{z}{2}\right| \leq 1$$

$$f_2(z) = \frac{1}{4+z^2} = \frac{1}{4} \cdot \frac{1}{\underbrace{1 + \left(\frac{z}{2}\right)^2}_{= 1 - \left(-\left(\frac{z}{2}\right)^2\right)}} = \frac{1}{4} \cdot \sum_{n=0}^{\infty} (-1)^n \left(\left(\frac{z}{2}\right)^2\right)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{4^{n+1}}$$

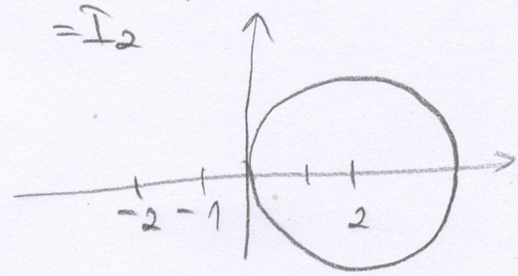
$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{-3^n}{2^{n+1}} \cdot \frac{1}{z^n} + \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{4^{n+1}}, \quad z \in D$$

5. Izračunajte:

$$I = \int_{|z-2|=2} \left(\frac{e^{z^3}\sin z}{(z+1)(z+2)} + (z+1)\sin\frac{1}{2z-3}\right)dz.$$

RJ. $$I = \underbrace{\int_{|z-2|=2} \frac{e^{z^3}\sin z}{(z+1)(z+2)}\,dz}_{f_1(z)} + \underbrace{\int_{|z-2|=2}(z+1)\sin\frac{1}{2z-3}\,dz}_{=I_2}$$

I_1



$$f_1(z) = \frac{e^{z^3}\sin z}{(z+1)(z+2)}$$

↳ analitička f-ja

osim u točkama -1 i -2,

ali one ne pripadaju području

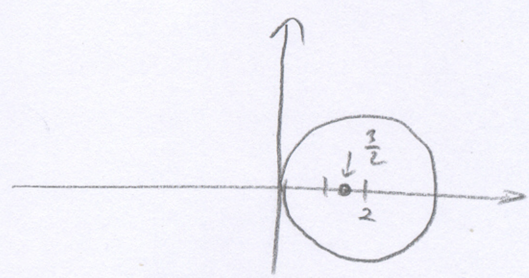
$|z-2| \leq 2$

$\Rightarrow f_1$ je analitička na $|z-2| \leq 2$

$\Rightarrow I_1 = 0$

$$f_2(z) = (z+1)\sin\frac{1}{2z-3}$$

singulariteti: $2z-3=0 \Leftrightarrow z = \frac{3}{2}$



$\rightarrow I_2 = 2\pi i \cdot \operatorname{Res} f_2\left(\frac{3}{2}\right)$

$$(z+1)\sin\frac{1}{2z-3} = \left\{\begin{array}{l} w = z - \frac{3}{2} \\ z = w + \frac{3}{2}\end{array}\right\}$$

$$= \left(w+\frac{3}{2}+1\right)\sin\frac{1}{2\left(w+\frac{3}{2}\right)-3} = \left(w+\frac{5}{2}\right)\sin\frac{1}{2w}$$

$$= \left(w+\frac{5}{2}\right)\sum_{n=0}^{\infty}\left(\frac{1}{2w}\right)^{2n+1}\cdot\frac{(-1)^n}{(2n+1)!} = \underbrace{w\cdot\sum_{n=0}^{\infty}\left(\frac{1}{2w}\right)^{2n+1}\frac{(-1)^n}{(2n+1)!}}_{\text{nema člana uz } w^{-1}} + \frac{5}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2w}\right)^{2n+1}\cdot\frac{(-1)^n}{(2n+1)!}$$

$$\Rightarrow \operatorname{Res} f_2\left(\frac{3}{2}\right) = \frac{5}{2}\cdot\frac{1}{2} = \frac{5}{4}$$

$$\Rightarrow I = I_1 + I_2 = 2\pi i\cdot\frac{5}{4} = \frac{5\pi i}{2}$$